And again divisively, $S - l : s :: r - \overline{r-1}\ (=1) : r$. Hence $r = \frac{s}{s-l}$.

Exa. $S = 9$, $l = 6$, and $r = \frac{9}{9-6} = 3$.

PROBLEM IV.

Having l and r to find $s - l$, without finding first s, (as might be done by *Probl.* 1)

RULE. $S - l = \frac{l}{r-1}$

DEMON. $S = \frac{r\,l}{r-1}$ hence $S \times \overline{r-1} = r\,l$, and $S : l :: r : r - 1$, and $s - l : l :: r - \overline{r-1}\ (=1) : r - 1$. Wherefore $S - l = \frac{l}{r-1}$

Exa. $l = 6$, $r = 3$, and $S - l = \frac{6}{2} = 3$

SCHOL. If the Ratio is $\frac{a}{b}$, then is $S - l = l \div \frac{a-b}{b} = \frac{b\,l}{a-b}$ for $\frac{a}{b} - 1 = \frac{a-b}{b}$

COROLLARIES.

1. If the Ratio is 2, then $s - l = l$; but in all other Cases $s - l$ is either greater or lesser than l. Reversely, if r is lesser than 2, or if $\overline{r-1}$ is lesser than 1, then is $l \div \overline{r-1}$ $(= s - l)$ greater than l. Again; if r is greater than 2, $r - 1$ is greater than 1; and hence $l \div r - 1$ $(= s - l)$ is less than l.

2. If the Ratio is a whole Number greater than 2, $s - l$ is such an aliquot Part of l, as $r - 1$ denominates; i.e. $S - l = \frac{1}{r-1}$ of l.

But *Observe* also, that $s - l$ can never be an aliquot Part of l, except when r is a whole Number greater than 2: It must be greater than 2, else $s - l$ is not less than l; as we saw in the last *Corollary*. Again; it must be an Integer: For suppose it a mix'd Number, as $2 + \frac{a}{n} = \frac{2n + a}{n}$ (which may represent any Number greater than 2, according as we take $\frac{a}{n}$) Then is $r - 1 = \frac{2n + a}{n} - 1 = \frac{n + a}{n}$. So that $s - l = l \div \overline{r-1} = l \div \frac{n+a}{n} = \frac{n}{n+a}$ of l. But if $\frac{n}{n+a}$ is an aliquot Part, then is n an aliquot Part of $n + a$; or n measures $n + a$, and because n measures n and $n + a$, therefore also it measures a; so that $\frac{a}{n}$ is an Integer; and $2 + \frac{a}{n}$ [the Ratio which makes $s - l$ an aliquot Part of l] is an Integer contrary to Supposition: Wherefore no mix'd Ratio can make a Series in which $s - l$ is an aliquot part of l.

3. If the Ratio is a mix'd Number $\frac{a}{b}$, whether greater or lesser than 2, $s - l$ is such a Fraction of l as $\frac{b}{a-b}$ expresses, i.e. $S - l = \frac{b}{a-b}$ of l; for, by the preceeding *Schol.* $S - l = \frac{b\,l}{a-b} = \frac{b}{a-b}$ of l. Whence again $s - l$ is a Multiple of l only in such Cases

wherein

wherein the loweſt Terms of the Ratio differ by 1. For ſuppoſe that is $\frac{a}{a-1}$ then is $s - l = \overline{a-1} + l$.

4. Hence again we learn another *Rule* for finding s by l and r; *viz.* firſt finding $s - l = \frac{l}{r-1}$, and then adding l to it, thus, $S = \frac{l}{r-1} + l$ (which added by the common Rules, comes to the former Rule $S = \frac{r\,l}{r-1}$)

Scholium. This *Problem* may be expreſs'd alſo in this manner, *viz.* of any Quantity l, let a certain Fraction, as $\frac{b}{a}$, be taken; then the ſame Fraction of this Fraction, and ſo on, continually taking the ſame Fraction of the preceeding; Then will the Sum of a Series of Quantities equal reſpectively to theſe ſeveral Fractions of l, be equal to $\frac{b}{a-b}$ of l; for the Series of the Fractions is $\frac{b}{a}$ of l : $\frac{b}{a}$ of $\frac{b}{a}$ of l, &c. or thus, $\frac{b\,l}{a}$, : $\frac{b^2\,l}{a^2}$ $\frac{b^3\,l}{a^3}$ &c. which is a Geometrical Series decreaſing in the Ratio $\frac{a}{b}$, whoſe Sum is therefore by the preceeding Rule, $\frac{b}{a-b}$ of l, or $\frac{b\,l}{a-b}$.

In what Caſes $\frac{b\,l}{a-b}$ is equal to l, or greater or leſſer, alſo when it is Multiple, or an aliquot Part of l, has been already explain'd.

PROBLEM V.

Having s and r to find $s - l$ (without finding firſt l, as we may do by *Problem* 2.)

Rule. $s - l = \frac{s}{r}$.

Demon. By *Probl.* 3d, $r = \frac{s}{s-l}$. Hence $r \times \overline{s-l} = s$, and $s - l = \frac{s}{r}$

Exa. $s = 9$, $r = 3$, and $s - l = \frac{9}{3} = 3$

Scholium. If we expreſs the Ratio thus, $\frac{a}{b}$, then $s - l = s \div \frac{a}{b} = \frac{s\,b}{a}$

COROLLARIES.

1. If r is an Integer, $s - l$ is an aliquot Part of s, *viz.* $\frac{1}{r}$ of s; but in all other Caſes $s - l$ is ſuch a proper Fraction of s as the Reciprocal of the Ratio expreſſes. So the Ratio being $\frac{a}{b}$, then $s - l = s \div \frac{a}{b} = \frac{b}{a}$ of s; for 'tis $\frac{b\,s}{a}$ by the *Schol.*

2. Hence we have another *Rule* for finding l by s and r; *viz.* Find $s - l = \frac{s}{r}$ and then ſubſtract $\frac{s}{r}$ from s, for $s - \overline{s-l} = l$; Wherefore $l = s - \frac{s}{r}$ which ſubſtracted, by the *Common Rules*, comes to the ſame expreſſion as before, *viz.* $\frac{s \times \overline{r-1}}{r}$

PROBLEM VI.

Having $r, s-l$ to find s, l, RULE. $s = r \times \overline{s-l}$. Then take $s-l$ from s, the Remainder is l; or find l independently of s, thus, $l = \overline{s-l} \times \overline{r-1}$.

DEMON. By the last $s - l = \frac{s}{r}$. Hence $s = r \times \overline{s-l}$. Again, by *Prob.* 4, $s - l = \frac{l}{r-1}$. Hence $l = \overline{s-l} \times \overline{r-1}$.

Exa. $r = 3$, $s - l = 3$: And $s = 3 \times 3 = 9$: Also $l = 3 \times 2 = 6$.

SCHOLIUM. If the Ratio is $\frac{a}{b}$, then is $s = \frac{\overline{s-l} \times a}{b}$. And, because $r - 1 = \frac{a-b}{b}$, hence $l = \frac{\overline{s-l} \times \overline{a-b}}{b}$.

COROL. If the Ratio is an Integer, then is s a Multiple of $s-l$; but the Ratio being a mix'd Number $\frac{a}{b}$, s is a mix'd Multiple of $s-l$, express'd by the Ratio; *that is*, $s = \frac{a}{b}$ of $s-l$; and $l = \frac{a-b}{b}$ of $s-l$.

As for the different Cases in which l is equal to $s-l$, or an aliquot Part or Multiple thereof, see *Corol.* 1, 2, 3. *Prob.* 4. where it's shewn in what Cases $s-l$ is equal to l, or is a Multiple, or aliquot Part thereof.

TABLE *of the preceeding* Problems.

Given	Sought	Solutions.		
r, l	s	$s = \frac{r.l}{r-1}$	Or supposing the Ratio $\frac{a}{b}$, or the 2d Term to be M, whereby the Ratio is $\frac{l}{M}$, then is.	$s = \frac{a}{a-b}$ of $l = \frac{l}{l-M}$.
r, s	l	$l = \frac{s \times \overline{r-1}}{r}$		$l = \frac{a-b}{a}$ of $s = \frac{\overline{l-M} \times s}{l}$
l, s	r	$r = \frac{s}{s-l}$		
l, r	$s-l$	$s - l = \frac{l}{r-1}$		$s - l = \frac{b}{a-b}$ of $l = \frac{M\ l}{l-M}$
s, r	$s-l$	$s - l = \frac{s}{r}$		$s - l = \frac{b}{a}$ of $s = \frac{M\ s}{l}$ $s = \frac{a}{b}$ of $s - l = \frac{l \times \overline{s-l}}{M}$
$r. s-l$	s, l	$\begin{cases} s = \overline{s-l} \times r \\ l = \overline{s-l} \times \overline{r-1} \end{cases}$		$l = \frac{a-b}{b}$ of $s - l = \frac{\overline{l-M} \times \overline{s-l}}{M}$

THEO-

THEOREM V.

A Series of Numbers may encreafe continually, yet fo that no Term of it can ever be fo great as a certain affignable Number.

Alfo a *Series* may decreafe continually, yet fo that no Term of it can ever be fo fmall as a certain affignable Number.

DEMON. 1°. Take an *Infinite Series* decreafing from any Number l, in any conftant Ratio r, as, $l : \frac{l}{r} . \frac{l}{r^2} . \frac{l}{r^3}$. Then take the Sums of this Series, repeated always from the beginning, thus, $l : l + \frac{l}{r} : l + \frac{l}{r} + \frac{l}{r^2} : l + \frac{l}{r} + \frac{l}{r^2} + \frac{l}{r^3} :$ &c. This is fuch an encreafing Series as was propos'd; for the Sum of the decreafing Series $l + \frac{l}{r}$ &c is limited to $\frac{rl}{r-1}$ (*Theor.* 4.) therefore no Term of the Series of its Sums can ever actually reach to $\frac{rl}{r-1}$.

2°. Take a decreafing Series $l : \frac{l}{r}$, &c. and fuppofe it fuch that all the Series wanting the firft Term is lefs than the firft Term; as it will be if r is greater than 2; (*Corol.* 1, *Prob.* 4.) then fubftract the fecond Term from the firft, and the third from the remainder, and fo on, and make a Series of thefe remainders, beginning with l, thus, $l : l - \frac{l}{r} . l - \frac{l}{r} - \frac{l}{r^2}$ &c. it is fuch a decreafing Series as was propos'd; for the Sum of the Series $l : \frac{l}{r}$: &c. wanting then the firft term l, *i. e.* the Sum of the Series $\frac{l}{r} : \frac{l}{r^2} . \frac{l}{r^3} :$ &c. is $\frac{l}{r-1}$ which is fuppos'd lefs than l; but fince that Series can never be exhaufted, it's plain, That no Term in the other Series, $l : l - \frac{l}{r}$ &c. can ever be fo fmall as the Difference betwixt l and $\frac{l}{r-1}$, becaufe we can never actually fubftract fo much from l as $\frac{l}{r-1}$, the Sum of the *Infinite Series* of Numbers fubftracted.

SCHOLIUMS.

1. We may take any Number A to begin a *Series*, and add to it fucceffively the Terms of any decreafing Series, $l : \frac{l}{r} . \frac{l}{r^2}$ &c. thus, $A : A + l : A + l + r$, &c. and this will be fuch an encreafing Series as was propos'd.

Again; we may take any Number A greater than the Sum of any decreafing Series $l : \frac{l}{r}$ &c. and fubftracting this out of A, make this Series $A : A - l : A - l - \frac{l}{r}$ &c. it will be fuch a decreafing *Series* as was propos'd. But to begin with the firft Term of the decreafing Series $l : \frac{l}{r}$ &c. makes a more regular *Series*.

2. Thefe *Infinite Series* encreafing or decreafing limitedly, are not, and cannot be, in a conftant Ratio; for then they could not be limited, by *Theor.* 1 & 2. And far lefs do the Ratio's of each Term to the preceeding encreafe; for then they would fo much fooner than with an equal Ratio outreach any propos'd Limitation: Wherefore 'tis plain, that of a *Series* encreafing for ever, but limitedly, the Series of its Ratio's, comparing each

Term to the preceeding, is a Series of Numbers which do continually decrease: and of a Series decreasing for ever, but limitedly; the Series of its Ratio's, comparing each Term to the preceeding, is a Series of Numbers which do continually encrease. But *observe* again, that tho' of a *Series* whose Terms encrease continually, but limitedly, the Ratios of each Term to the preceeding do continually decrease, yet the Reverse will not hold; that is, tho' a Series of encreasing Numbers is such that the Ratio's of each Term to the preceeding continually decrease, yet it does not follow that the *Series* encreases under a limitation; which one Example demonstrates: For if we take any Encreasing Arithmetical Progression, as 1, 2, 3, 4, *&c* its Ratio's, comparing each Term to the preceeding, are $\frac{2}{1} : \frac{3}{2} : \frac{4}{3}$ *&c.* which do continually decrease, and will do so in all such Cases (as has been shewn *Cor. Theor.* 17, Ch. 5, B. 4.) yet we can find a Term in the *Series* greater than any assignable Number. The like is also true as to a decreasing Series, *viz.* That they may decrease by encreasing Ratio's of each Term to the preceeding, and yet be unlimited in their decrease; of which we have Examples, by taking the reciprocal Ratio's of any *Arithmetical Series*, and making a *Series* decreasing according to these Ratio's; as in this *Example*, $1 : \frac{1}{2} : \frac{1}{3} : \frac{1}{4} : \frac{1}{5}$, *&c.* where the Ratio's of each Term to the preceeding are 1 : 2, 2 : 3, 3 : 4.

COROL. A *Series* may decrease continually, and yet have a Sum infinitely great: For if it decreases Limitedly, then its Sum will be always greater than as many Terms equal each to the Limiting Number, since none of the Terms can ever be so little as that Number.

THEOREM VI.

Take any Geometrical Progression encreasing from 1, which may be universally represented, thus, $1 : r : r^2 : r^3 :$ &c. then

1°. Take the Sums of this *Series* continually from the beginning, thus, $1 : 1 + r : 1 + r + r^2 :$ &c, and divide each of these Sums by the last Term added in each, thus, $1 : \frac{1+r}{r} : \frac{1+r+r^2}{r^2}$ &c. and this is a *Series* encreasing limitedly, so as no Term can ever reach to $\frac{r}{r-1}$.

2°. Instead of the Sums, substract from each Term of the ∺ *l* (*that is*, from each Power of r) all the preceeding Terms, and make this Series $1 : \frac{r-1}{r} : \frac{r^2-r-1}{r^2} : \frac{r^3-r^2-r-1}{r^3}$: &c. 'Tis a *Series* decreasing limitedly, in which no Term can ever reach to $\frac{r-2}{r-1}$.

3°. Take the *Series* of the Reciprocals of the first Series, *viz.* take $1 : \frac{r}{1+r} : \frac{r^2}{1+r+r^2}$: &c. 'tis a *Series* decreasing limitedly, in which no Term can ever reach to $\frac{r-1}{r}$.

4° Take

4°. Take the Series of the Reciprocals of the second Series, *viz.* take $1 : \frac{r}{r-1} : \frac{r^2-r-1}{r^2}$: &c. 'tis a Series encreasing limitedly, in which no Term can ever reach to $\frac{r-1}{r-2}$.

DEMON. 1°. For the first Series, $1 : \frac{1+r}{r} : \frac{1+r+r^2}{r^2}$: &c. 'tis no other than the sums of this decreasing Series, $1 : \frac{1}{r} : \frac{1}{r^2}$ &c. added by the Common Rules, thus, $1+\frac{1}{r}=\frac{1+r}{r}$ and $\frac{1+r}{r}+\frac{1}{r^2}=\frac{1+r+r^2}{r^2}$: But the sum of this Series $1 : \frac{1}{r} : \frac{1}{r^2}$ &c. is $\frac{r}{r-1}$. Therefore,

2°. For the second Series, $1 : \frac{r-1}{r} : \frac{r^2-r-1}{r^2}$: &c. it is no other than the Effect of taking every Term after the first of this Series, $1 ; \frac{1}{r} ; \frac{1}{r^2}$ &c. out of the first, and out of the succeeding Remainders; for $1-\frac{1}{r}=\frac{r-1}{r}$, and $\frac{r-1}{r}-\frac{1}{r^2}=\frac{r^2-r-1}{r^2}$ &c. and the Sum of the Series $\frac{1}{r} . \frac{1}{r^2} : \frac{1}{r^3}$: &c. is $\frac{1}{r-1}$, which therefore can never be all taken away from 1; whence it's plain, that no Term of this *Series* of *Remainders* can ever reach so low as $1-\frac{1}{r-1}=\frac{r-2}{r-1}$. And *Observe*, that we have in this Conclusion a plain Demonstration that the Numerators in each Term of the Series, $1 : \frac{r-1}{r} : \frac{r^2-r-1}{r^2}$ &c. are positive Numbers, *i. e.* that any Power of r (a whole or mix'd Number) is greater than the Sum of all the preceeding Terms, $+1$; for the Sum of the whole Series, $\frac{1}{r} : \frac{1}{r^2}$ &c. is $\frac{1}{r-1}$, which is less than 1, and therefore each Term of the Series $1 : \frac{r-1}{r}$ &c. is a real positive Number; which it cannot be, unless the Numerator is a positive Number, *i. e.* unless r^2 is greater than all the preceeding Terms. But this Truth may be also demonstrated from the Rules of a Geometrical Progression $1 . r . r^2$, &c.

3°. For the third Series, $1 : \frac{r}{1+r} : \frac{r^2}{1+r+r^2}$ &c. it decreases limitedly, so as never to reach to $\frac{r-1}{r}$; because if it did reach that Number, then 'tis evident that the Reciprocal Series $1 : \frac{1+r}{r} : \frac{1+r+r^2}{r^2}$ must reach to the reciprocal Number $\frac{r}{r-1}$; which it can never do by Article the First.

4°. For the fourth Series, $1 : \frac{r}{r-1} ; \frac{r^2}{r^2-r-1}$ &c. because 'tis the Reciprocal of the second, which can never decrease to $\frac{r-2}{r-1}$, therefore this encreases so, as it can never reach to $\frac{r-1}{r-2}$; for if it could, the other would decrease to $\frac{r-2}{r-1}$, which is shewn to be impossible in Article second.

SCHO-

SCHOLIUMS.

1. If we compare the two Encreasing Series, $1 : \frac{1+r}{r}$ &c. and $1 : \frac{r}{r-1}$ &c. then each Term of the first (after 1) is lesser than the Corresponding Term of the second; thus $\frac{1+r}{r}$ is lesser than $\frac{r}{r-1}$; and so on. To *Demonstrate* which universally, let any Number A be Denominator of a Fraction, and $A+B$ the Numerator, thus, $\frac{A+B}{A}$; also make A Numerator, and $A-B$ the Denominator, thus, $\frac{A}{A-B}$; then is $\frac{A+B}{A}$ less than $\frac{A}{A-B}$: For if we reduce them to a common Denominator, they are $\frac{A^2-B^2}{A^2-AB}$ $\left(=\frac{A+B}{A}\right)$ and $\frac{A^2}{A^2-AB}$ $\left(=\frac{A}{A-B}\right)$; whereby it's plain the first is less than the second, because A^2-B^2 is less than A^2: But A may represent any Power of r, as r^n, and B the Sum of all the preceeding Terms of the *Series*, (which Sum is always less than r^n); so that $\frac{A+B}{A}$ may represent any Term of the Series $1 : \frac{1+r}{r}$: &c. and $\frac{A}{A-B}$ any Term of the Series $1 : \frac{r}{r-1}$: &c. which finishes the Demonstration.

Hence the Value of the first Series is less than that of the other, (*i. e.* the Value of any assignable Number of Terms of the one compar'd to as many of the other.)

2d. Compare the two Decreasing Series $1 : \frac{r}{1+r}$ &c. and $1 : \frac{r-1}{r}$ &c. and each Term of the first (after the 1) is greater than the Corresponding of the other, *viz.* $\frac{r}{1+r}$ greater than $\frac{r-1}{r}$ &c. because the Reciprocal of that is lesser than of this; *i. e.* $\frac{1+r}{r}$ less than $\frac{r}{r-1}$ (from the nature of Fractions). Or we may demonstrate this the same Way as the former; for $\frac{A}{A+B}$ is greater than $\frac{A-B}{A}$, because being reduced to one Denominator, they are $\frac{A^2}{A^2+AB}=\frac{A}{A+B}$ and $\frac{A^2-B^2}{A^2+AB}=\frac{A-B}{A}$: Hence the Value of the first is greater than that of the other.

3d. If we take the Ratios of these several *Series*, comparing each Term to the preceeding, they make these regular Series:

$$\frac{1+r}{r} : \frac{1+r+r^2}{r+r^2} : \frac{1+r+r^2+r^3}{r+r^2+r^3} \text{ \&c. for the Series, } 1 : \frac{1+r}{r} : \frac{1+r+r^2}{r^2} \text{ \&c}$$

$$\frac{r-1}{r} : \frac{r^2-r-1}{r^2-r} : \frac{r^3-r^2-r-1}{r^3-r^2-r} \text{ \&c. for the Series, } 1 : \frac{r-1}{r} : \frac{r^2-r-r}{r^2} \text{ \&c}$$

$$\frac{r}{1+r} : \frac{r+r^2}{1+r+r^2} : \frac{r+r^2+r^3}{1+r+r^2+r^3} \text{ \&c. for the Series, } 1 : \frac{r}{1+r} : \frac{r^2}{1+r+r^2} \text{ \&c}$$

$$\frac{r}{r-1} : \frac{r^2-r}{r^2-r-1} : \frac{r^3-r^2-r}{r^3-r^2-r-1} \text{ \&c. for the Series, } 1 : \frac{r}{r-1} : \frac{r^2}{r^2-r-1} \text{ \&c}$$

Where

Where you see the third and fourth are Reciprocals of the first and second, because the Series of which they are the Ratios, are so.

THEOREM VII.

Let there be two *Infinite Series* of Numbers $\div l$; And let their corresponding Terms be multiplied together, *i. e.* the first Term of the one by the first of the other, and so on; The products make an *Infinite Series*, $\div l$ whose Sum is in some cases Infinite, in others Finite. Thus,

1°. If each Series consists of Terms equal among themselves, or the one having equal Terms, and the other encreasing; or, lastly, both encreasing, the Sum of the Product is Infinite.

2°. If both Series decrease, or if the one Series has equal terms, and the other Decreases, the sum of the products is Finite; notwithstanding the Sum of the Series of equal Terms is Infinite.

3°. If the one encreases and the other decreases, the Sum of the Products is in some cases Infinite, and in some Finite, notwithstanding the Sum of the encreasing one be always Infinite: particularly, if the Ratio of the encreasing Series is equal to or lesser than the Reciprocal Ratio of the decreasing one, the sum of the products is Infinite; but if it's greater, the Sum is Finite.

DEMON. By *Theorem* 3. *Ch.* 4, *B* 4. The Series of Products is $\div l$ in the Ratio compounded of those of the Series Multiplied. So the Ratio of the one Series is $\frac{A}{B}$, and of the other it is $\frac{L}{M}$, whose product is $\frac{AL}{BM}$ the Ratio of the Series of Products; And because that Series either consists of equal terms, or it encreases or decreases in a constant Ratio, therefore it will accordingly have either a Finite or Infinite Sum: So that what remains to be shewn is only the Correspondence of the several Cases, in which it encreases or decreases or is equal, to the Theorem; which will easily appear thus,

$$\left.\begin{array}{l} A : B : C : D : \&c. \\ L : M : N : O : \&c. \\ \hline AL : BM : CN : DO : \&c. \end{array}\right\}$$

1°. If both the Series encrease, or both consist of equal Terms, or the one equal, and the other encreasing, it's evident the Sum of any one of them is Infinite (by *Theorem* 3.) and much more is the Series of their Products so. Or thus, the two Ratio's $\frac{A}{B}$, $\frac{L}{M}$ are either both proper Fractions, or the one so, and the other equal to 1, or both equal to 1: Wherefore $\frac{AL}{BM}$ is either a proper Fraction, so that the Series of Products encreases in that constant Ratio, and the Sum is Infinite (*Theorem* 3.); or it is equal to 1, and so the Products are equal, and here also the Sum is Infinite, (*Theorem* 3.)

2°. If both the Series decrease, or one decreases, and the other equal; then $\frac{A}{B}$ and $\frac{L}{M}$ are improper Fractions, and hence $\frac{AL}{BM}$ is an improper Fraction: Therefore the Series of Products decreases in that constant Ratio, and so the Sum is Finite (*Theorem* 4.)

3°. If the one Series encreases, and the other decreases; We shall suppose that $A : B : C :$ encreases, so that $\frac{A}{B}$ is a proper Fraction; and that $L : M : N :$ &c. decreases, so that $\frac{L}{M}$ is an improper Fraction: Now the Ratio of the compound Series $AL : BM$, &c.

is

is $\frac{AL}{BM}$, which, I say, is a proper Fraction when $\frac{A}{B}$ is less than $\frac{M}{L}$, but is an improper Fraction when $\frac{A}{B}$ is equal to, or greater than $\frac{M}{L}$; for these Fractions reduced to one Denominator, are $\frac{AL}{BL}\left(=\frac{A}{B}\right)$ and $\frac{MB}{LB}\left(=\frac{M}{L}\right)$ Wherefore, as AL is equal to, greater or lesser than MB, consequently $\frac{AL}{BM}$ (the Ratio of the Compound Series) is either $=1$; if $AL=BM$ (*i.e.* if $\frac{A}{B}=\frac{M}{L}$) and in this Case the Compound Series consists of equal Terms, and so the Sum is Infinite: Or $\frac{AL}{BM}$ is a proper Fraction, if AL is less than BM (*i.e.* if $\frac{A}{B}$ is less than $\frac{M}{L}$) and then the Series encreases, so that the Sum is here also Infinite: Or, lastly, $\frac{AL}{BM}$ is an improper Fraction, if AL is greater than BM, (i. e. $\frac{A}{B}$ greater than $\frac{M}{L}$) and so the *Series* decreases, and the Sum is consequently *Finite*.

Scholiums.

1. If we suppose each Term of the one *Series* is multiply'd into each Term of the other, the Sum of the Products will be *Infinite* in all Cases, except when the two *Series* do both decrease: for the Sum of all these Products is the Products of the Sums of the two *Series*; and if but any one of them is Encreasing, or Equal, its Sum is *Infinite*; Which therefore multiply'd into the other Sum, whether *Finite* or *Infinite*, must make an *Infinite Product*: but both the *Series* decreasing, their Sums are *Finite*, and consequently their Product is *Finite*.

2. Tho' we have only suppos'd *Series* encreasing or decreasing in one constant Ratio, yet we may consider other Kinds of *Series*: As, *first*, we may suppose any Kind of *Series* whose Terms continually Encrease, whether in one constant Ratio, or not; and any such *Series* being put instead of one which is $\div l$, in the first Article of the *Theorem*, the Conclusion will be the same, as is most evident, tho' the *Compound Series* will not be $\div l$. Again, in the second Article of the *Theorem*, we may suppose a *Series* decreasing limitedly, instead of one in $\div l$; and if two such be multiply'd together, or one such with a Series of equal Terms, the *Compound Series* will certainly have an Infinite Sum; because each Term of this suppos'd Series being greater than the Limiting Number, must have a greater Effect than a Series of equal Terms equal to that Number; but such a limited Series multiply'd into a Series decreasing in one Ratio, will have a Finite Sum, because the Series decreasing limitedly will have a lesser Effect than a Series of equal Terms equal to the first Term of this Series, which would make a Finite Sum in the Series of Products. Again, suppose two Series decreasing in Ratio's that do also decrease, (comparing each Term to the preceeding) the Sum of the Products is Finite; which it is also if one of the Series is such, and the other equal or decreasing limitedly. In the third Article of the *Theorem* let us suppose, first, a Series encreasing so, that the Reciprocal Ratio's do continually encrease, (and so the Ratio's themselves decrease) and another decreasing by a Constant Ratio; then, if the first Ratio of the Encreasing Series is equal to, or less than the Reciprocal of the decreasing one, the Sum of the Products is certainly Infinite; for they will make a Series encreasing, and whose reciprocal Ratio's will also encrease, because each Ratio of that

that *Series* will be gradually less and less than the reciprocal Ratio of the other: But tho' the first Ratio of the encreasing Series is greater than the reciprocal Ratio of the decreasing one, it will not be true that the Sum of the Products is Finite, unless all the following Ratios of the encreasing Series be also greater. Again, *secondly*, suppose a Series encreasing in a Constant Ratio, and another decreasing, so that the reciprocal Ratios do decrease, (and so the Ratio's themselves encrease) then if the Ratio of the encreasing Series is greater than the Reciprocal of the first Ratio of the decreasing one, it will be greater than all the following ones; and consequently the Series of Products will decrease in Ratios whose Reciprocals decrease, and so the Sum will be Finite. And if the Ratio of the Encreasing Series be equal to the first Reciprocal Ratio of the other, then it will be greater than all the following ones (which decrease); and here again the Sum of the Products will be Finite. But tho' the Ratio of the Encreasing Series be less than the first reciprocal Ratio of the other, the sum of Products is not Infinite, unless it be also less than each of the rest of the reciprocal Ratios of the other Series; for it may become equal to some one of them, or greater, and then the Sum of the Product will be Finite. *Thirdly*, If the one *Series* encreases by Ratios that decrease (or whose Reciprocals encrease) and the other by Ratios whose Reciprocals decrease, then if each Ratio of the first Series is equal to, or less than, the reciprocal Ratio of the Correspondent Terms of the second Series, the sum of the Products is Infinite; but if greater, the Sum is Finite. *Fourthly*, If the one Series encreases in whatever manner, and the other decreases limitedly, the Sum of the Products will be Infinite; because this decreasing Series is greater than a Series of equal Terms, equal to the limiting Number.

THEOREM VIII.

If from any quantity A we take away any proper Fraction of it, as $\frac{a}{b}$; and then of what remains take away the same Fraction, and so on continually; the Sum of the *Infinite Series* of Parts taken will be equal to the whole A, and the Sum of the *Infinite Series* of the Parts left at every Substraction is equal to $\frac{b-a}{a}$ of A.

$$\text{Taken.}\quad \frac{aA}{b} : \frac{a\times\overline{b-a}\times A}{b} : \frac{a\times\overline{b-a}^2\times A}{b\times b^2} : \frac{a\times\overline{b-a}^3\times A}{b\times b} : \&c$$

$$\text{Left.}\quad \frac{\overline{b-a}\times A}{b} : \frac{\overline{b-a}^2\times A}{b^2} : \frac{\overline{b-a}^3\times A}{b^3} : \frac{\overline{b-a}^4\times A}{b^4} : \&c$$

DEMON. The Series of the Parts *taken* and *left* are evidently these, express'd above; for the first Part taken being $\frac{a}{b}$ of A $\left(=\frac{aA}{b}\right)$ what remains is $\frac{b-a}{b}$ of A, $\left(=\frac{\overline{b-a}\times A}{b}\right)$ for $\frac{a}{b}+\frac{b-a}{b}=\frac{b}{b}=1$. Then of this Remainder we take again $\frac{a}{b}$ Parts, which makes $\frac{a}{b}$ of $\frac{b-a}{b}$ of A $\left(=\frac{a\times\overline{b-a}\times A}{b\times b}\right)$ and there remains plainly the $\frac{b-a}{b}$ Parts of $\frac{\overline{b-a}\times A}{b}$, which is $\frac{\overline{b-a}^2\times A}{b^2}$: Whereby it's manifest, that every Series left is the $\frac{b-a}{b}$ Parts of the preceeding; and every Term of the Series taken is

 the

the $\frac{a}{b}$ Parts of the preceding Term of the Series left, or also the $\frac{b-a}{b}$ Parts of the preceding Term of the same Series. *Now* these two Series proceed, each decreasing in a constant Ratio, which is the same in both, *viz.* $\frac{b}{b-a}$; for every Term being $\frac{b-a}{b}$ of the preceeding, the Ratio of that preceeding to the following is the reciprocal $\frac{b}{b-a}$, and applying the Rule of *Probl.* 1, the Sum of the Series taken is A; thus, $\frac{aA}{b} \times \frac{b}{b-a} = \frac{aA}{b-a}$. And this divided by $\frac{b}{b-a} - 1$, or $\frac{a}{b-a}$ quotes A. Again; the Sum of the Series left is $\frac{b-a}{a}$ of A; for $\frac{b-a \times A}{b} \times \frac{b}{b-a} = A$, which divided by $\frac{b}{b-a} - 1 = \frac{a}{b-a}$ gives $\frac{b-a}{a}$ of A.

Observe also, that the Series of Parts left being demonstrated to be an *Infinite Decreasing Series*, the other may be deduced from it, thus: Since the Series of Parts left may be carried on till there be a Term less than any assignable Quantity, hence it plainly follows, that the Sum of the Parts taken away shall want less of the whole A than any assignable Difference; which is all that's meant by saying, That that Sum is equal to A.

SCHOLIUMS.

1°. If the Correspondent Terms of these two *Series* of the Parts taken and left are multiply'd together, the Sum of the Compound Series is *Finite*, and equal to $\frac{b-a}{2b-a}$ of A^2; which is easily prov'd from the common Rules, thus: The common Ratio of both the *Series* is $\frac{b}{b-a}$, therefore the Ratio of the Compound Series is $\frac{b^2}{\overline{b-a}^2} = \frac{b^2}{b^2 - 2ab + a^2}$; but the first Term of the one Series is $\frac{a}{b}$ of A, and of the other it is $\frac{b-a}{b}$ of A. Hence the first Term of the Compound Series is the Product of these, *viz.* $\frac{a \times \overline{b-a}}{b^2}$ of A^2, or $\frac{ab - a^2}{b^2}$ of A^2. And, according to *Probl.* 1, we must multiply this by the Ratio $\frac{b^2}{b^2 - 2ab + a^2}$ the Product is $\frac{ab - a^2}{b^2 - 2ab + a^2}$ of A^2; which again divided by the Ratio less Unity, *viz.* $\frac{b^2}{b^2 - 2ab + a^2} - 1 = \frac{2ab - a^2}{b^2 - 2ab + a^2}$, the Quote is $\frac{ab - a^2}{2ab - a^2}$ of $A^2 = \frac{b-a}{2b-a}$ of A^2.

2°. If we suppose each Term of the one Series multiply'd into each of the other, the Sum of the Products is Finite, and it is particularly equal to $\frac{b-a}{a}$ of A^2: For the Sum of Parts taken away is A, and the Sum of the Parts left is $\frac{b-a}{a}$ of A, and their Product is $\frac{b-a}{a}$ of A^2.

3°. Hence

3°. Hence we have these Proportions, or Ratios of the total Values of these several Series *viz.* (1°.) the Sum of the parts taken is to the Sum of the remainders *viz.* $A : \frac{b-a}{a}$ of A, as $a : b-a$. (2°.) the Sum of the Series taken, to the Sum of the Products of the correspondent Terms of the two *viz.* $A : \frac{b-a}{2b-a}$ of A^2, as $1 : \frac{b-a}{2b-a}$ of A: Or $2b-a : \overline{b-a} \times A$. (3°.) The Sum of the Series taken, is to the Sum of the Products of all the Terms of the one Series, multiplied into all the Terms of the other, *viz.* $A : \frac{b-a}{a}$ of A^2, as $1 : \frac{b-a}{a}$ of A, or $a : \overline{b-a} \times A$.
(4°) The Sum of the Series of Parts left, is to the Sum of the first Products, *viz.* $\frac{b-a}{a}$ of $A : \frac{b-a}{2b-a}$ of A^2, as $\frac{b-a}{a} : \frac{b-a}{2b-a}$ of A. (5°.) The Sum of the Parts left, is to the Sum of the second Products, *viz.* $\frac{b-a}{a}$ of $A : \frac{b-a}{a}$ of A^2 as $1 : A$
(6°.) The two Sums of Products, *viz.* $\frac{b-a}{2b-a}$ of $A^2 : \frac{b-a}{a}$ of A^2, as $a : 2b-a$.

THEOREM IX.

In the Arithmetical Progression 1, 2, 3, 4, &c. The Sum is to the Product of the last Term by the Number of Terms, *i. e.* to the Square of the last Term; in a Ratio always greater than that of 1, 2. But approaching infinitely near to it.

DEMON. The Sum of the Arithmetical Progression is $\frac{n^2+n}{2}$ *Probl.* 5. *Chap.* 2. *B.* 4.) And the Square of the last Term n is n^2, therefore the Sum is to that Square as $\frac{n^2+n}{n} : n^2 :: n^2+n : 2n^2 :: n+1 : 2n$. But $\frac{n+1}{2n} = \frac{n}{n2} + \frac{1}{n2} = \frac{1}{2} + \frac{1}{2n}$ And as the Number of Terms, or last Term n encreases, so does $\frac{1}{2n}$ decrease infinitely, therefore the Ratio approaches infinitely near to $\frac{1}{2}$.

Observe if the Arithmetical Series begins with 0, thus, 0, 1, 2, 3, then the Sum is to the Product of the last Term by the Number of Terms, exactly in every Step, as 1 to 2, for the Sum is in this Case, $\frac{n^2-n}{2}$. But the last Term is $n-1$, and its Product by the Number of Terms n is n^2-n, therefore the Sum is to the Product as $\frac{n^2-n}{2} : n^2-n :: n^2-n : 2n^2-2n :: 1 : 2$.

THEOREM X.

Take the natural Progression beginning with 0, thus, 0, 1, 2, 3, &c. And take the Series of any the like powers of the former Series; As the Squares, 0, 1, 4, 9, &c. Or Cubes, 0, 1, 8, 27, &c. Then again take the sum of the Series of Powers to any Number of Terms, and also multiply the last of the Terms summed by the Number of Terms, (reckoning always 0 for the first Term.) The *Ratio* of that *Sum* to that Product is more than $\frac{1}{n+1}$ (n being the Index of the Powers) *i. e.* in the Series of Squares

it is more than $\frac{1}{3}$; in the Cubes more than $\frac{1}{4}$; and so on: But the Series going on *in infinitum*, we may take in more and more Terms without end into the Sum; and the more we take, the Ratio of the *Sum* to the *Product* mentioned grows less and less; yet so as it can never actually be equal to $\frac{1}{n+1}$ but approaches infinitely near to it, or within less than any assignable difference.

DEMON. The truth of this Theorem has hitherto, that I know of, been demonstrated only by an induction; or shewing that it is true in Squares, Cubes, and a few more where an actual examination of it has been made; and then its concluded that since it holds true in every Case where it has been actually tried, and no reason appearing against this being an universal Rule or Law in the nature of Numbers, therefore it is true in all other Cases. It must be acknowledged that where we find the same general Law observed in a Variety of Cases of different Powers, taken at Pleasure, as in the second, third, fourth, the eighth, the thirteenth, the twentieth, and many more taken up and down among the infinite Variety of Powers; we have great Reason to believe that it's a general Law in all Cases, tho' we don't see a direct and positive Reason or Demonstration for it; yet it's as certain that this is but an imperfect Proof, or a probability of its being true. *Again*, as to the Demonstrations given of the particular Cases whence the general Theorem is deduced, they are also of the same nature, *i. e.* they are taken only from seeing the thing proposed, to be true in as many Cases as have been actually tried, *i. e.* the Series of any Power being summed to 2 Terms, or 3, or 4 it's found to be always true, that the Ratio of the Sum to the Product mentioned is more than $\frac{1}{n+1}$ but still diminishing as the number of Terms becomes greater; and diminishing also in such a certain, constant Tenor, as shews that if it proceed so, it will approach infinitely near $\frac{1}{n+1}$. Now after having premised this, concerning the Method that has hitherto satisfied for the Demonstration of this Theorem, I shall shew how it appears in a few particular Cases, and how it may by the same Method be investigated in any other Cases at pleasure.

Example in Squares.

Arithmetical *Series*.	0 .	1 .	2 .	3 .	4. &c.	
Squares.	0 .	1 .	4 .	9 .	16. &c.	

$$\text{Sum of two Terms. / Products.}\quad \frac{0+1}{2\times 1}=\frac{1}{2}=\frac{1}{3}+\frac{1}{6}$$

$$\text{Sum of three Terms. / Products.}\quad \frac{1+4}{3\times 4}=\frac{5}{12}=\frac{1}{3}+\frac{1}{12}$$

$$\text{Sum of four Terms. / Products.}\quad \frac{5+9}{4\times 9}=\frac{14}{36}=\frac{1}{3}+\frac{1}{18}$$

$$\text{Sum of five Terms. / Products.}\quad \frac{14+16}{5\times 16}=\frac{30}{80}=\frac{1}{3}+\frac{1}{24}$$

And so for a Series of *Squares*.

By taking the Sums, and the Product of the last Term summed, multiplied by the Number of Terms; the Ratio is in all Cases equal to $\frac{1}{3}$ + a certain aliquot Fraction, which goes on decreasing by a constant addition of 6 (the Denominator of the first of these Fractions) to the Denominator of the preceding; which Fractions, because this Law is to be constantly observed, do therefore decrease infinitely; so that the Ratio of the Sum to the Product approaches infinitely near to $\frac{1}{3}$ by the infinite decreasing of the Fraction which is still actually joined with $\frac{1}{3}$ in every step.

Exa.

Exa. for *Cubes.*

Arithm. Progress.	$0 . 1 . 2 . 3 . 4$ &c.
Cubes.	$0 . 1 . 8 . 27 . 64$ &c.
Sum of 2 Terms. / Prod.	$\frac{0+1}{2\times1}=\frac{1}{4}=\frac{1}{4}+\frac{1}{4}$
3 Terms. / Prod.	$\frac{1+8}{3\times8}=\frac{9}{24}=\frac{1}{4}+\frac{1}{8}$
4 Terms. / Prod.	$\frac{9+27}{4\times27}=\frac{36}{108}=\frac{1}{4}+\frac{1}{12}$
5 Terms. / Prod.	$\frac{36+64}{5\times64}=\frac{100}{320}=\frac{1}{4}+\frac{1}{16}$ &c &c

2°. For *Cubes.* The Sums and Products being taken and compar'd, as in the Margin, the Ratio is continually $\frac{1}{4}$ + an aliquot Fraction, which goes on decreasing by a constant addition of 4 (the Denominator of the first of these Fractions) to the Denominator of the preceding; which Law being constantly observ'd, that Fraction join'd to $\frac{1}{4}$ becomes infinitely little; *i. e.* the Ratio becomes infinitely near to $\frac{1}{4}$.

SCHOLIUMS.

I. If we examine the superior Powers, the same General Truth will be found in them; but the Fractions adhering to the $\frac{1}{n+1}$ will not diminish in the same manner as they have been observ'd to do in the Squares and Cubes; *i. e.* by a constant addition of the Denominator of the first of these Fractions to the preceding Denominator: Yet this I have found, as far as I have examin'd them, that these Fractions decrease so, that their Ratios to one another, comparing each to the preceding, do also constantly decrease; which makes the Fractions themselves decrease so much quicker than if they decreas'd in one constant Ratio; Or if they decreas'd by Encreasing Ratios, as these in the Squares and Cubes do; which you'll easily observe do so decrease, that their Ratios are the Series of the Reciprocal Ratios of the natural Progression $1 : 2 : 3 : 4$. For $\frac{1}{6} : \frac{1}{12} : : 2 : 1$, and $\frac{1}{12} : \frac{1}{18} : : 3 : 2$, also $\frac{1}{18} : \frac{1}{24} : : 4 : 3$. and so also in the Cubes, $\frac{1}{4} : \frac{1}{8} : : 2 : 1$, and $\frac{1}{8} : \frac{1}{12} : : 3 : 2$, and $\frac{1}{12} : \frac{1}{16} : : 4 : 3$. And it's observable here too, that the first of these Fractions, in the *Squares* being $\frac{1}{6}$, and in the *Cubes* $\frac{1}{4}$, they decrease faster in those than in these; *i. e.* at the same distance from the beginning the Ratio is nearer equal to $\frac{1}{3}$ in the *Squares*, than it is to $\frac{1}{4}$ in the *Cubes*. But, in the 4th and 5th Powers, and others which I have actually examin'd, they decrease faster, because of their decreasing Ratios.

Another Way of proposing the Theorem.

II. Some Authors (particularly *Sturmius*) have propos'd this *Theorem* in another View, thus: They make the Arithmetical Progression begin with 1, and assert the same Truth concerning the Ratios of the Sums and Products, *viz.* that it approaches Infinitely near to $\frac{1}{n+1}$: But any I have met with give us no other kind of Demonstration than what has been given above, *i. e.* by *Induction*, or arguing from a few Particulars: And because their Method of investigating these particular Cases is somewhat different from the preceding, I shall here explain it.

1°. For

1°. For Squares.

		Sums.	Products.		
		1 . 2 . 3 . 4 . &c.			
		1 . 4 . 9 . 16 . &c.			
For 3 Terms.	14	:	27	::	$1 + \frac{1}{2} + \frac{1}{18} : 3$
For 6 Terms.	91	:	216	::	$1 + \frac{1}{4} + \frac{1}{72} : 3$

In the *Squares* take the first 3 Terms, and at every succeeding Step twice as many, and the Ratio of the *Sum* to the *Product* is found always greater than 1 : 3. But so as to decrease, and become infinitely near to it; because the two Fractions that adhere to 1 (in the lesser Term of the Ratio) are found to decrease in a Constant Ratio, which is $\frac{1}{2}$ in the first Fraction, and $\frac{1}{4}$ in the second Fraction.

Whence *Sturmy* concludes the Argument in this manner, *viz.* Since of the two Fractions adhering to 1 at every Step, the first is always $\frac{1}{2}$ of that in the preceding Step, and the second is $\frac{1}{4}$ of that in the preceding Step; therefore these two Fractions are in every Step the Effect of substracting from $\frac{1}{2} + \frac{1}{18}$, which belongs to the first Step, this Series, $\frac{1}{4} + \frac{3}{72} : \frac{1}{8} + \frac{3}{288}$, making 3 the Numerator of the second Part, because $\frac{1}{18} = \frac{4}{72}$, and $\frac{4}{72} - \frac{3}{72} = \frac{1}{72}$, and so of the rest: But this Series consisting of two Series, take their Sums separately, they are $\frac{1}{4} + \frac{1}{8} + \frac{1}{16}$: &c. $= \frac{1}{2}$: and $\frac{3}{72} + \frac{3}{288}$ &c. $= \frac{1}{18}$, by *Probl.* 1. that Series being suppos'd to be Infinite; whence, in the last Case, or when the Series of Squares is infinite, the Fractions adhering to 1 are evanish'd, because they are become $\frac{1}{2} + \frac{1}{18} - \frac{1}{2} - \frac{1}{18} = 0$.

As to the last part of this Demonstration, I *observe*, that it is superfluous; for the Argument ought to be concluded immediately from this, That the Fractions adhering to 1 decrease in one Constant Ratio: For, in this Case, if we suppose the Series *infinite*, these Fractions must decrease to nothing, this being the very Supposition upon which the Rule is founded, by which we find the Sum of an *Infinite decreasing Series*: And this Rule being used in the Argument to prove that these Fractions do at last evanish, it's manifest that the Thing to be concluded is already suppos'd.

2°. For *Cubes*.

		Sums.	Products.		
		1 . 2 . 3 . 4 : &c.			
		1 . 8 . 27 . 64 : &c.			
For 4 Terms.	100	:	256	::	$1 + \frac{1}{2} + \frac{1}{16} : 4$
For 8 Terms.	1296	:	4096	::	$1 + \frac{1}{4} + \frac{1}{64} : 4$

Again; in the *Cubes* take the first 4 Terms, and then 8 Terms, and so on doubling the Number; and hereby 'tis found that the Ratio of the Sum to the Product is always greater than 1 : 4. but approaching Infinitely near to it, because the two Fractions adhering to 1 (in the lesser Term of the Ratio) do decrease in a constant Ratio, *viz.* $\frac{1}{2}$ in the first, and $\frac{1}{4}$ in the second Fraction.

Sturmy concludes the Argument here the same Way as in the Case of *Squares*, to which the same Observation already made is also applicable.

So far then you see an Agreement in this Form of the *Theorem*, and the preceeding *viz.* that in the *Squares* the Ratio of the Sum and Product decreases faster, because $\frac{1}{18}$ is less than $\frac{1}{16}$, and the Ratios of these decreasing Fractions equal in both; just as in the preceding Form the Ratio also decreas'd faster in the *Squares* than *Cubes*, and by the same Ratios.

III. But again; Tho' I have found no better Way of demonstrating the *Theorem* (in either of the two Views of it explain'd) as to its universal extension to all the different Powers, yet as to the Demonstration of the first two particular Cases (which are the most useful) viz. *Squares* and *Cubes*, 1 being the first Term, I shall here shew you a new and easie

easie Demonstration, deduced directly *à priori* from the Canons given in *Ch.* 2, for the summing the *Squares* and *Cubes* of the Arithmetical Progression.

A Direct Demonstration *of the preceding* Theorem *for* Squares *and* Cubes, *supposing the Series to begin with* 1.

(1°) For *Squares.* The Sum of the *Squares* of an Arithmetical Progression 1. 2. 3. 4, supposing the Number of Terms to be n, is $\frac{2n^3+3n^2+n}{6}$ (*Prob.*3,*Ch.*2) and in the Arithmetical Progression the last Term is always the Number of Terms, therefore the last Term of the Series of *Squares* is n^2; which multiply'd by the Number of Terms n, the Product is n^3: Wherefore the Ratio of the Sum to the Product is $\frac{2n^3+3n^2+n}{6} : n^3 :: 2n^3+3n^2+n : 6n^3 :: 2n^2+3n+1 : 6n^2$ (by dividing each Member by n) but $\frac{2n^2+3n+1}{6n^2} = \frac{2n}{6n^2}+\frac{3n}{6n^2}+\frac{1}{6n^2} = \frac{1}{3}+\frac{1}{2n}+\frac{1}{6n^2}$ $\left(\text{for } \frac{2n^2}{6n^2} = \frac{1}{3} \text{ and } \frac{3n}{6n^2} = \frac{1}{2n}\right)$. Again; it's obvious that the greater n is, the less will these Fractions $\frac{1}{2n}, \frac{1}{6n^2}$ be; and that they will decrease so, as to become infinitly little, or less than any assignable Quantity: Therefore the Ratio of the Sum to the Product, tho' it's always greater than $\frac{1}{3}$ (being for the first two Terms $\frac{1}{3}+\frac{1}{4}+\frac{1}{24}$.) yet the Fractions adhering to $\frac{1}{3}$ decreasing (as now demonstrated) so as to become infinitely small, the Ratio approaches infinitely near to $\frac{1}{3}$.

(2°) For *Cubes.* The Sum of the *Cubes* of the Arithmetical Progression is $\frac{n^4+2n^3+n^2}{4}$ (*Probl.* 3, *Ch.* 2.) and the Product of the last Term n^3, by the Number of Terms, n is n^4: So that the Ratio of the Sum to the Product is $\frac{n^4+2n^3+n^2}{4} : n^4 :: n^4+2n^3+n^2 : 4n^4 :: n^2+2n+1 : 4n^2$. And $\frac{n^2+2n+1}{4n^2} = \frac{n^2}{4n^2}+\frac{2n}{4n^2}+\frac{1}{4n^2} = \frac{1}{4}+\frac{1}{2n}+\frac{1}{4n^2}$. And because the greater that n is, the less will $\frac{1}{2n}$ and $\frac{1}{4n^2}$ be; therefore, tho' the Ratio of the Sum to the Product is always greater than $\frac{1}{4}$ (being for two Terms, $\frac{1}{4}+\frac{1}{4}+\frac{1}{16}$) yet the Fractions adhering to the $\frac{1}{4}$ decreasing infinitely, the Ratio approaches infinitely near to $\frac{1}{4}$.

THEOREM XI.

Take the Series of the *Squares* of the natural Progression 1. 2. 3. &c. *viz.* 1. 4. 9. 16. &c. multiply any Term of this Series by the number of Terms from 1; from the Product substract the sum of all the preceeding lesser *Squares* (which Sum will always be a lesser Number than that Product) the Ratio of the Difference to the Product is always greater than 2 : 3. But approaching infinitely near to it, the farther the Series is carried, or the greater the *Square* assumed is.

Observe:

Observe. Since the Number of Terms from 1 to any Term in the Series of *Squares* is the Root of that *Square*: therefore the Product of the *Square* by the Number of Terms is the *Cube* of the same Root; and therefore the *Theorem* may be propos'd thus: The Ratio of the Difference betwixt the *Cube* of any Integral Number and the Sum of the *Squares* of all Numbers less than that, to that *Cube*, is greater than 2 : 3, but approaching infinitely near to it, as we chuse that *Cube* greater and greater.

Exa. 1. The *Cube* of 6 is 216, and the *Squares* of 1. 2. 3. 4. 5. are 1. 4. 9. 16. 25, whose Sum is 55, then is $216 - 55 = 161$. And $\frac{161}{216} = \frac{2}{3} + \frac{51}{648}$; as you'll find by taking $\frac{2}{3}$ out of $\frac{161}{216}$.

2. The *Cube* of 7 is 343, and the *Squares* of 1. 2. 3. 4. 5. 6. are 1. 4. 9. 16. 25. 36, whose Sum is 91; then $343 - 91 = 252$, and $\frac{252}{343} = \frac{2}{3} + \frac{70}{1029}$, and this Fraction $\frac{70}{1029}$ is less than $\frac{51}{648}$

DEMON. 1°. Let any Number be represented by $n + 1$, its *Cube* will be $n^3 + n^2 + 3n + 1$; and if we take all the Numbers lesser than $n + 1$, they make the Arithmetical Progression 1. 2. 3, &c. whose last Term (and number of Terms) is n; and the Sum of their *Square* is $\frac{2n^3 + 3n^2 + n}{6}$, which is manifestly a lesser Number than the *Cube* of $n + 1$, or $n^3 + 3n^2 + 3n + 1$.

2°. Take the Difference propos'd, *viz.* $n^3 + 3n^2 + 3n + 1 - \frac{2n^3 + 3n^2 + n}{6}$
$$= \frac{6n^3 + 18n^2 + 18n + 6}{6} - \frac{2n^3 + 3n^2 + n}{6} = \frac{6n^3 + 18n^2 + 18n + 6 - 2n^3 - 3n^2 - n}{6}$$
$$= \frac{4n^3 + 15n^2 + 17n + 6}{6}$$
Now compare this Difference to $\overline{n + 1}^3$, or, $n^3 + 3n^2 + 3n + 1$, the Ratio is plainly $4n^3 + 15n^2 + 17n + 6 : 6n^3 + 18n^2 + 18n + 6$, or, $\frac{4n^3 + 15n^2 + 17n + 6}{6n^3 + 18n^2 + 18n + 6}$, which is greater than $\frac{2}{3}$ by this Fraction $\frac{9n^2 + 15n + 6}{18n^3 + 54n^2 + 54n + 18}$, as you'll find by substracting $\frac{2}{3}$ out of it, by the common Rules. Now since in every Step the Ratio of the Difference to the *Cube* will be the same way express'd, 'tis plain it will always be greater than $\frac{2}{3}$; but if the Fraction $\frac{9n^2 + 15n + 6}{18n^3 + 54n^2 + 54n + 18}$, adhering to it, grows infinitely little, or less than any assignable Fraction, then the Ratio approaches infinitely near to $\frac{2}{3}$. What remains then to be demonstrated, is only this Infinite decrease of the Fraction adhering to the $\frac{2}{3}$ which is thus done: Were the adhering Fraction $\frac{9n^2}{18n^3}$, its infinite decrease is easily shewn, for it is equal to $\frac{1}{2n}$, by dividing Numerator and Denominator equally by $9n^2$, but n is gradually taken equal to 1. 2. 3. 4, &c. and therefore $\frac{1}{2n}$ is gradually $\frac{1}{2}$. $\frac{1}{4}$: $\frac{1}{6}$. $\frac{1}{8}$, &c. which manifestly decreases infinitely, or so as to become less than any assignable Fraction. Again;

Again; $\frac{9n^2+15n+6}{18n^3+54n^2+54n+18}$ is a Fraction less than $\frac{9n^2}{18n^3}$ or $\frac{1}{2n}$, as is easily seen by the Comparison; wherefore if $\frac{1}{2n}$ does decrease infinitely as n encreases, the other which is less than $\frac{1}{2n}$ must also decrease infinitely.

COROLL. The Sum of the *Squares* of the Series 1 . 2 . 3. &c. carried to any Number of Terms, is to the Difference betwixt the Cube of the Number of Terms, or last Term, and the Sum of all the *Squares*, except the last, in a Ratio approaching infinitely near to 1 : 2. but still greater: For the Ratio of the Sum of the Squares to the Cube of the Number of Terms approaches infinitely near to $\frac{1}{3}$, but still greater (by *Theorem* 9.) And by the present *Theorem* the Ratio of the Difference mention'd, to that Cube, approaches infinitely near to $\frac{2}{3}$; therefore the Ratio of the Sum of the Squares to the Difference approaches infinitely near to $\frac{1}{2}$

SCHOL. This *Theorem* is propos'd by some Authors (particularly *Sturmius*) in a very different manner, which is to this purpose, *viz.* If any Series (of Integers) begins with a square Number, and decreases by growing Differences, which are the Series of odd Numbers, 1. 3. 5, &c. the Sum of the Series carried down to o (in which 'twill always end) is to the Product of that *Square* (which is the greatest Term) by the Number of Terms, in a Ratio always greater than $\frac{2}{3}$, but approaching infinitely near to it, as we take that *Square* greater.

I shall first shew the Coincidence of this and the preceding Proposition, and then give you *Sturmius*'s demonstration.

For the first, take the odd Series 1. 3. 5, &c, the Sum of it to any Number of Terms is the *Square* of the Number of Terms: or the Series of its Sums taken always from the beginning, makes the Series of square Numbers. (*Corol.* 4, *Probl.* 5, *Ch.* 2, *B.* 4.)

1	3	5	7	9	&c.	odd Numbers.
1	4	9	16	25	&c.	square Numbers.
1	2	3	4	5	&c.	Roots.

Wherefore it is plain, that if from any square Number we take successively as many Terms of the Series of odd Numbers from 1, as the Root of that *Square* expresses, when the last Substraction is made, there remains nothing; and that *Square*, with the several Remainders, is the Series propos'd. Thus, beginning with 9, it is 9, 8, 5; beginning with 25, it is 25, 24, 21, 16, 9: Universally, if it begin with n^2, it is n^2; n^2-1; n^2-1-3; $n^2-1-3-5$; &c. which again is the same as n^2; n^2-1; n^2-4; n^2-9; &c. the Series being carried to as many Terms as the Root n expresses; And the *Square* substracted from n^2 in the last Term, being that next lesser than n^2, because 'tis the Sum of a Number of Terms of the odd Series less by 1 than n.

Whence the Coincidence of the two Propositions is evident; for the Number of Terms being n, therefore the Sum of this Series is equal to the Difference of n^3 and $1+4+9$ &c. carried to a Number of Terms equal to $n-1$, and n^2 multiply'd by the Number of Terms n, is n^3.

2°. The Demonſtration that *Sturmy* gives us of this *Theorem* is only by Induction, in the ſame manner as in the preceding *Theorems*, thus: If there are three Terms, 9, 8, 5, whoſe Sum is 22, then is $22 : 27\ (= 3 \times 9) :: 2 + \frac{4}{9} : 3$, which is a greater Ratio than that of $2 : 3$. Again; $\frac{4}{9}$ being $= \frac{1}{2} - \frac{1}{18}$, he expreſſes it thus, $2 + \frac{1}{2} - \frac{1}{18} : 3$. If there are ſix Terms, $36 : 35 : 32 : 27 : 20 : 11$; their Sum is 161; and, $161 : 216\ (= 6 \times 36) : 2 + \frac{19}{72} : 3$; Or, as, $2 + \frac{1}{4} - \frac{1}{72} : 3$. So he goes on examining more Caſes, taking at every Step double the number of Terms of the laſt Step; and finds that the Fraction adhering to 2 proceeds in a continued Geometrical Progreſſion decreaſing, thus, $\frac{1}{2} - \frac{1}{18}, \frac{1}{4} - \frac{1}{72}, \frac{1}{8} - \frac{1}{288}$: the firſt part decreaſing in the Ratio of 2 to 1, and the other in the Ratio of 4 to 1; whence he concludes the Argument in the ſame manner as in *Theorem* X; to which is applicable the ſame Obſervation I made upon that.

Obſerve: As to this laſt Method of propoſing the *Theorem*, That it is accommodated to a particular Uſe which *Sturmy* makes of it in Geometry: The Reaſon I choſe the other Way being, That in this Shape I found the direct Demonſtration I have given of it.

CHAP. IV.

Of Infinite Decimals.

WHAT a *Decimal Fraction* is, and its Notation; alſo what a *Circulating Decimal* is, with the whole Operations about *Determinate Decimals*, has all been already taught: But, that the whole Doctrine of *Infinite Decimals* may be found here together, ſome of theſe Definitions muſt be repeated.

DEFINITIONS.

I. A Decimal Fraction may be call'd *Finite* or *Determinate*, when it has certain and determinate Numbers for its Numerator and Denominator; *i. e.* when the Numerator and Denominator have each a certain limited Number of Figures, as, $.3 = \frac{3}{10}$, $.046 = \frac{46}{1000}$.

II. A Decimal Fraction may be call'd *Infinite* or *Indeterminate* when the Number of Places is *Indeterminate*, and encreaſing without End; whereby the Numerator and Denominator are conceiv'd to be themſelves infinitely great. So in this *Exa.* .347 *&c.* if we ſuppoſe that there ought to be more and more Figures, *in infinitum*, annex'd to the right hand of thoſe here ſet down, thereby encreaſing both Numerator and Denominator infinitely, or without end, we do hereby form the Idea of an Infinite and Indeterminate Decimal.

III. *Infinite Decimals* are of two kinds, which we may diſtinguiſh by the General Denominations of *Certain* and *Incertain*,

A *Certain* Infinite Decimal is ſuch whoſe Numerator runs into Infinity by a continual repetition of one or more Figures; as in theſe Examples, .44,&c. .033,&c. .455,&c. wherein the ſame Figures, 4, 3, 5 is conſtantly repeated: Alſo .356356,&c. where 356 is repeated; and .c7236464, &c. where 64 are conſtantly repeated. Such Decimals are alſo particularly call'd *Repeating* or *Circulating Decimals*, from this continual repetition or circulation of the ſame Figures in the Numerator. *Obſerve* alſo, that the Figure or Figures repeated may very conveniently and properly be call'd the *Repetend*.

Incertain Decimals are ſuch whoſe Numerator goes on for ever without a conſtant circulation of Figures.

SCHOLIUMS.

1*ſt*. The eſſential Difference betwixt theſe two kinds of *Infinite Decimals* is this; that the *Certain* have a determinate, finite, and certain Value; *i. e.* that there is a certain determinate Vulgar Fraction, which expreſſes the true and compleat Value of that Infinite Decimal (as ſhall be demonſtrated) whereas the *Incertain* have no ſuch finite and aſſignable Value: And this is the reaſon of theſe Names.

Now 'tis owing to this Certain finite Value of a Circulating Decimal that they occur in Practice; for they are no other thing than the Reſult or Effect of reducing ſome Vulgar Fraction to a Decimal: Or if they are brought into a Queſtion by meer Suppoſition, yet they are reducible to a Vulgar finite Fraction; as all Vulgar Fractions are reducible into a Decimal, either Finite or Circulating (as will be explain'd afterwards). It is this Property alſo which makes theſe Decimals, tho' they cannot be limited in that Decimal form, capable of ſuch management in Practice, as that no Error can happen from the impoſſibility of finiſhing and determining it in form.

2*d*. As to the *Incertain* Decimals, *obſerve*, that tho' they have no determinate value, yet their value is not Infinite; for notwithſtanding it encreaſes without end, yet (as ſhall be ſhewn) they are ſo limited in general, as that tho' the whole Infinity could be actually exhauſted, yet the value of the Fraction cannot exceed ſome finite aſſignable Fraction, as on the other hand it cannot come ſhort of ſome ſuch Fraction; wherefore we may juſtly ſay, they have a finite, tho' not an aſſignable value; for if their value were aſſignable, they would circulate contrary to Suppoſition. But then alſo *obſerve*, that becauſe their value is not aſſignable, there is no poſſibility of ſupplying their Defects perfectly, ſo that we muſt be content to do it nearly, or by way of Approximation.

Again *obſerve*, that tho' no *Incertain* Decimal can ever ariſe from any finite aſſign'd Fraction, yet they are not all ſo imaginary, but that they do in ſome manner, and in ſome Caſes, neceſſarily occur in Practice of *Arithmetick*; as in the Extraction of Surd Roots.

Alſo, ſometimes where we might have a Circulating Decimal, it may be convenient to take it imperfect; as when in the reduction of a Vulgar Fraction to a Decimal, the Diviſion has gone far on without coming to a Circulation; then we may chuſe to take it with a convenient number of Places, and conſider it as *Incertain*.

The *laſt Remark* I make here is, That ſome Decimals are Incertain as to their value, becauſe they do not circulate; yet, in another reſpect, they may be ſaid to be Certain; for we may conceive a Numerator conſiſting of certain known Figures, ſucceeding one another in a certain order for ever, yet ſo as there be no Circulation. *Exa.* .34 344 3444, &c. ſuppoſing that after 3 you have 4 repeated once more than in the preceding Step; or this, .34 345 3456, &c. beginning ſtill at 3, and after it taking the natural ſeries, in order to one place more in every ſtep. But now, tho' ſuch Decimals ſhould in any Caſe occur, (which yet ſeem to be meerly imaginary) this Certainty of the Figures of the Numerator does not make the Value certain; and ſo we know no better how to manage them than if

there were no such Certainty. *Again*; As to these Incertain Decimals which express Surd Roots, tho' they are not always of this kind, (and I don't know if any of them was ever found to be so, or if they can be) yet they have a Certainty of another kind, which is, that the Progress of the Fraction depends upon a certain Law or Rule, whereby the Extraction is perform'd (as has been explain'd in its place): And, because of this, they are not purely imaginary and supposititious, yet, as to the Practice, they have no more Accuracy than the rest, except in the Case of raising them to the Reciprocal Power express'd by the Denominator of the Root, when 'tis known of what Number they express the Root, and in some Comparisons of them among themselves; but then these Comparisons are manag'd not by means of the Decimal Expressions, but of the Powers themselves with their Surd or Radical Indexes; as explain'd in *Book* III.

IV. *Circulating Decimals* (or *Circulates*, as they may be conveniently call'd) are distinguish'd into *Pure* and *Mix'd*. 1°. *Pure*, when there are no significant Figures in the Numerator, but what belong to the Repetend; *i. e.* when there are no Figures, or none but 0's, betwixt the Point and the Repetend; as, .3434 &c, .0044 &c, .046046 &c, .00303 &c.

2°. *Mix'd*, when there are significant Figures betwixt the Point and the Repetend; as, .344 &c, .304848 &c, .0407272 &c. Which I call *Mix'd* because they consist of two parts, a *Finite* and a *Circulate*. So the first *Exa.* is plainly the Sum of these, .3 + .044 &c, the second is .30 + .004848 &c, the third is .046 + .0007272 &c.

And here also *Observe*, that as Decimals are more generally distinguish'd into *Simple* and *Mix'd*, so, when we name a *mix'd* Decimal, it signifies a Decimal with an Integer; but a *mix'd* Circulate more strictly regards only the Decimal part: Yet, to avoid too many Distinctions, when a whole Number is join'd with a Circulate, whether *pure* or *mix'd*, we may call it a *mix'd Circulate*; as this, 46.33 &c. or this, 238.0044 &c. And as any Integer may be reduced to a Fractional Expression with a Decimal Denominator, so the finite part of all mix'd Circulates may be express'd fractionally, by taking for Numerator all the Figures as they stand, from the highest place of the Integral part to the Repetend, and for the Numerator, 1 with as many 0's as there are Places betwixt the Point and the Repetend, thus, 46.03232 &c. $= \frac{460}{10} +$ .032 &c. also, 374.235858 &c. $= \frac{37423}{100} +$ .0058 &c. which in the proper Decimal form are 374.23 + .0058 &c. for tho' 374.23 has, in this form, the Integral and Fractional Parts distinguish'd, yet they express decimally an Improper Fraction, when all the Figures, neglecting the Point, are made Numerator; as has been explain'd in its place.

V. *Circulates* whose Repetends consist of the same number of Figures, and begin also at the same place after the Point, may be call'd *Like* or *Similar Circulates*, whether they be both *pure* or *mix'd*, or one *pure*, and the other *mix'd*: So these are *Similar*, .33 &c. and .77 &c. and these, 2.345656 &c. .004242 &c.

Scholiums.

1. If the Repetend be twice written down with an &c. after it, this will clearly shew that there is a repetition, and what the Repetend is: But this we may do more conveniently, by setting a Point over the first and last Figure of the Repetend once written down: Thus instead of .033 &c. write $.0\dot{3}$; And for .4376376 &c. write $.4\dot{3}7\dot{6}$; and so of others.

Again;

2*d.* Becauſe of the different Views in which a *Circulate* may be taken, 'twill be convenient to call the Repetend which is abſolutely the firſt in any Circulate, *The Given Repetend*; ſo here $.3434$ &c. 34 is the Given Repetend. But as the ſame Fraction may be conſider'd in another view, *viz.* as a mix'd Circulate equal to $.3 + .04\dot{3}$ (as you'll find preſently explain'd) ſo the Repetend it has in this view may be call'd for diſtinction the *New Repetend*: Alſo the Finite Part preceding the Given Repetend is the Given Finite Part, and that preceding the New Repetend is the New Finite Part.

3*d.* The Circulation of a Decimal may begin in the Integral Part; as, $\dot{3}.\dot{3}$ or $\dot{4}.2\dot{3}$, or $\dot{3}4.05\dot{7}$: Now if there are no Figures but what belong to the Repetend, it's in that reſpect a *pure* Circulate: But as we have limited that Name to the Fractional Part by it ſelf, we ſhall leave this other kind of Circulate to the Claſs of *mix'd* Circulates, as a particular Species of it; ſo that we muſt reckon the firſt Period of the Repetend, as it belongs to the Fractional part, to be that which begins firſt after the Point, thus, $\dot{4}.2\dot{3} = 4.\dot{2}342\dot{3}$; or rather $4.\dot{2}3\dot{4}$; ſee *Theor.* 3.

THEOREM I.

Any Finite Decimal may be conſider'd as Infinite, by annexing 0's without end on the right hand of the Numerator, making 0 the Repetend; thus, $.34 = .3400$, &c. or $.34\dot{0}$.

DEMON. The Numerator and Denominator of the Given Fraction being equally multiply'd by the 0's join'd *in infinitum*, the value of the Fraction is ſtill the ſame.

THEOREM 2.

Any *Pure Circulate* may be conſider'd as *Mix'd*, and keep ſtill the ſame Repetend; by taking the given Repetend once or oftner written down for a Finite Part; and conſidering the ſame Repetend as circulating after that for the Infinite Part; thus, $.\dot{3}\dot{4} = .34 + .00\dot{3}\dot{4} = .3434 + .0000\dot{3}\dot{4}$.

DEMON. The Reaſon is obvious, ſince as far as an Infinite Fraction is continued, ſo far the Value is finite and determinate; and the remaining part is ſtill infinite, tho' of a leſs value than the given Infinite, becauſe of what's determin'd and taken away.

THEOREM 3.

If any *Circulate* has a Repetend of more than one Figure, it may be transform'd into another Circulate having a Repetend of the ſame number of Figures, and alſo the ſame Figures, but in another order, *i.e.* by beginning a new Repetend from any Figure after the firſt of the given Repetend; and that taken either in the firſt or ſecond, or any other Period of the given Repetend, leaving all the Figures on the left of this new Repetend to the Finite part; whereby if the given Circulate was *pure*, it will in ſome caſes become *mix'd*; or if it was *mix'd*, the Finite part becomes always greater, and the Infinite leſs; thus, $.\dot{3}\dot{4} = .3\dot{4}\dot{3} = .34\dot{3}\dot{4}$; $.\dot{4}56\dot{7} = .4\dot{5}67\dot{5} = .4567\dot{5}67\dot{5}6$, $.00\dot{4}\dot{2} = .004\dot{2}\dot{0} = .00\dot{4}20\dot{0}$.

DEMON. In the firſt Example, ſince 34 is ſuppos'd to be repeated for ever, if 3 is taken away, there muſt remain $.04\dot{3}$; or, if 34 is taken away, there remains $.0\dot{3}\dot{4}$; Since 3 ſucceeds 4, and 4 ſucceeds 3 for ever. The ſame Reaſon is obvious in every Caſe; which you'll alſo find afterwards further confirm'd.

COROLL.

COROLL. Any Circulate may be transform'd into another, whose Repetend begins at any distance after the given Finite Part.

SCHOL. If the Repetend of a *pure Circulate* has o's in the first places, on the left hand; then, whether it begins immediately after the Point, or have o's betwixt it and the Point, it's manifest that the changing of the Repetend, in the manner here explain'd, will not make it a *Mix'd Circulate*, if the new Repetend begins at the first significant Figure, or at any of these o's, in the first Period of the given Repetend; but if it begin at any Figure after the first significant Figure of the first Period, or at any Figure in any of the other Periods, it will be a *Mix'd Circulate*; thus, $.\dot{0}04\dot{6} = .\dot{0}046\dot{0}$ (a *Pure Circulate*) $= .00\dot{4}60\dot{0}$ (a *Pure Circulate*) $= .004\dot{6}00\dot{4}$ (a *Mix'd Circulate*).

In any other kind of *pure Circulates*, a new Repetend will certainly make it a *Mix'd Circulate*.

THEOREM 4.

Any *Circulate* may be transform'd into another having a greater Repetend, *i. e.* one having more Places; by taking the given Repetend, or any of equal number of Places into which it is transform'd by the last, as oft as we please, and considering all that as a New Repetend, thus, $.\dot{4} = .\dot{4}\dot{4} = .\dot{4}4\dot{4}$; Also $.0\dot{4}\dot{2} = .0\dot{4}24\dot{2}$; And $.0\dot{3}6\dot{4} = .0\dot{3}6436\dot{4} = .036\dot{4}3643\dot{6}$

The Reason of this is obvious.

Observe; When we speak of the Repetend of a Circulate without distinction, it's always to be understood of the least Repetend.

THEOREM 5.

Any two or more Circulates may be made Similar, by making all the Repetends begin where that one of them begins which stands farthest from the Point (by the Method explain'd in *Theor.* 3, and *Coroll.*) And, to make them end together, let each of them have as many Places as the number of Units express'd by the least common Multiple of the several Numbers of Places in all the given Repetends (and, to find that least common Multiple, see *Probl.* 5, *Ch.* 1, *B.* 5.) So, in both the annex'd Examples, that least common Multiple is *6*.

Ex. 1. $.4\dot{3}\dot{6} = .43\dot{6}3636\dot{3}$

$.04\dot{7} = .04\dot{7}7777\dot{7}$

$.29\dot{3}4\dot{1} = .29\dot{3}4134\dot{1}$

Ex. 2. $.426\dot{7} = .426\dot{7}7777\dot{7}$

$4.9\dot{3}\dot{2} = 4.932\dot{3}2323\dot{2}$

$26.\dot{3}2\dot{8} = 26.328\dot{3}2832\dot{8}$

DEMON. The Reason of that Part concerning the Beginning of the new Repetends is plain from *Theor.* 3. And as to their ending together, it's plain, that if they are all repeated so oft as that the number of Places taken in each is a common Multiple of the several numbers of Places in each given Repetend; Then, as that may be all taken for a new Repetend, (by *Theor.* 4.) so it will make them all end together, and be consequently Similar. And, lastly, the Reason why we take their least common Multiple, is to have the Expression as short and neat as possible.

THEO-

THEOREM 6.

Every Circulate has a finite assignable Value, thus :

Part 1st. If it's a *Pure* Circulate, it's equal to a Vulgar Fraction whose Numerator is the Repetend, and its Denominator a Number express'd by as many 9's as there are Places in the Repetend, with as many 0's on the right hand as there are 0's betwixt the Point and Repetend.

Exa. (1) $.\dot{3} = \frac{3}{9}$ (2d) $.0\dot{4} = \frac{4}{90}$ (3d) $.\dot{4}\dot{6} = \frac{46}{99}$ (4th) $.00\dot{3}7\dot{2} = \frac{372}{99900}$ (5th) $.\dot{0}04\dot{6} = \frac{0046}{9999}$ or $\frac{46}{9999}$

Universally. Let R express the Repetend, and a the number of 0's betwixt the Point and Repetend, the Sum is $\frac{R}{9, \&c.}$ or $\frac{R}{9, \&c. \times 10^a}$ (10^a expressing that power of 10 whose Index is a.)

Part 2d. If it's a *Mix'd Circulate*, find the Sum of the Circulating Part, and add it to the Finite Part ; which total Sum being express'd all together as a simple Fraction, will have for Denominator that of the Finite Value of the Circulating Part ; and for Numerator the Sum of these two Numbers, *viz.* the Repetend, and the Product of the Numerator of the Finite Part (express'd fractionally) by the same Denominator, without the 0's, if any belong to it ; *i. e.* by a Number of 9's as many as there are Places in the Repetend.

Exa. (1) $.46\dot{3} = \frac{46 \times 9 + 3}{900}$ (2d) $46.0\dot{2}\dot{7} = \frac{460 \times 99 + 27}{990}$

(3d) $8.327\dot{4}6\dot{1} = \frac{8327 \times 999 + 461}{999000}$ (4th) $3.\dot{4}\dot{6} = \frac{3 \times 99 + 46}{99}$

Universally. Let A be the Numerator of the Finite Part, and 10^a its Denominator, the Sum is $\frac{A \times 9 \&c. + R}{9 \&c. \times 10^a} = \frac{B}{9 \&c. \times 10^a}$ (taking $B = A \times 9 \&c. + R$.)

And *Observe*, That as the multiplying by 9 *&c.* is a very easy Operation, [See *Case* 6, § 2d, *Ch.* 5, *B* 1.] so the Multiplication, and Addition of R to the Product, may be done all at once very easily, thus : Subtract the first right-hand Figure of the Numerator A from the first of the Repetend R ; and so on in Order thus ; In the preceding *Exa.* 1, the Operation is 6 from 3 I cannot take, but from 13, and 7 remains; then 5 from 6, and 1 remains ; lastly, 0 from 4, and 4 remains; and the result is $417 = 46 \times 9 + 3$. In *Ex.* 2d it is $7 - 0 = 7$. $12 - 6 = 6$. $10 - 5 = 5$. $6 - 1 = 5$. $4 - 0 = 4$. the result being $45567 = 460 \times 99 + 27$. In *Exa.* 3d it is $11 - 7 = 4$. $6 - 3 = 3$. $4 - 3 = 1$. $17 - 8 = 9$. $2 - 1 = 1$. $83 - 0 = 83$. the result being $8319134 = 8327 \times 999 + 461$. In *Exam.* 4th it is $6 - 3 = 3$. $34 - 0 = 34$. making $343 = 3 \times 99 + 46$. But had this last Example been $9.\dot{4}\dot{6}$, it were $16 - 9 = 7$. $94 - 1 = 93$, making $937 = 9 \times 99 + 46$. These Examples sufficiently illustrate the Practice. And, to make it clearer, do these Examples at large ; first multiplying by 9 *&c.* by the Method of the Rule refer'd to, then add the Repetend.

DEMON. For the First Part. Every *Pure Circulate* is, from the nature of a Decimal Fraction, a Series of Decreasing Fractions whose Numerators are all the same, *viz.* the given Repetend, and their Denominators are a Geometrical Series encreasing in the constant Ratio express'd by 1, with as many 0's as (*i.e.* whose Ratio is a Power of 10, having for its Index) the number of Places in the Repetend (taking here the Ratio as the Quote of the greater Term divided by the lesser) thus, $.33 \&c. = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000}$ *&c.* the Ratio of the Denominators being 1 : 10 ; Also $.0004646, \&c. = \frac{46}{100000} +$

+ 46

+ $\frac{24}{1000000}$ &c. the Ratio of the Denominators being 1 : 100. Again, .0404 &c. $= \frac{04}{100} + \frac{04}{10000}$ &c. or $\frac{4}{100} + \frac{4}{10000}$ &c. But Fractions having a common Numerator are in the Ratio of their Denominators reciprocally; wherefore the several Terms or Finite Decimals, of which a Circulate is compos'd, make an Infinite decreasing Geometrical Series, whose common Ratio is the Ratio of their Denominators, which may be express'd universally 10^m, supposing as many o's as the Repetend has Figures, or m to be equal to the number of Places in the Repetend. Again: Let R represent the Repetend or common Numerator of this Series of Fractions, and 10^n the Denominator of the first Fraction, which therefore is $\frac{R}{10^n}$; then, by the Rules of Infinite Series, the Sum is $\frac{R}{10^n} + 10^m \times \overline{10^m - 1}$ [for l being the greatest $l \times l^2$, and r the Ratio, the Sum is $rl \div \overline{r-1}$]. Now in the Dividend $\frac{R}{10^n} \times 10^m$, the Multiplier 10^m being the Ratio, it's manifest it cannot have more Places than 10^n, the Denominator of the greatest Extreme; but it may have fewer, or the same Number. If it have the same, *i. e.* if $n = m$, then is $\frac{R}{10^n} \times 10^m$ (or 10^n) $= R$; but if n is greater than m, 'tis evident that 10^n must have as many more o's as the number of o's from the Point to the Repetend. Therefore in this Case the Product $\frac{R}{10^n} + 10^m$ may be simply express'd $\frac{R}{10^{n-m}}$ ($n - m$ expressing the number of o's from the Point to the Repetend). And if we take $a = n - m$, it is $\frac{R}{10^a}$. Then for the Divisor $10^m - 1$, it's plainly $= 9$ &c. taking as many 9's as there are o's in 10^m (for $10 - 1 = 9$; $100 - 1 = 99$, and so on). From all which it is clear, that the Sum is universally $\frac{R}{9\text{\&c}}$ if there is no o betwixt the Point and Repetend; but if there is, then the Number of them being $= a$, the Sum is $\frac{R}{9\text{ \&c.} \times 10^a}$

For the 2d Part; Let A express the Numerator of the Finite Part, and 10^a its Denominator, which Part is therefore $\frac{A}{10^a}$; then the Circulating Part being $\frac{R}{9\text{ \&c.} \times 10^a}$, the Sum is $\frac{A}{10^a} + \frac{R}{9\text{ \&c.} \times 10^a}$, which by due reduction is $\frac{A \times 9\text{ \&c.} + R}{9\text{ \&c.} \times 10^a} = \frac{B}{9\text{ \&c.} \times 10^a}$ (taking $B = A \times 9$ &c. $+ R$.)

COROLLARIES.

1. If the Repetend of any Circulate is 9, the Value or Sum of that Series is an Unit of the Place next that Repetend on the left hand, so, $.\dot{9} = 1$, $.0\dot{9} = .1$ $.00\dot{9} = .01$, and so on. The Reason is plain from the *Theorem*; for $.\dot{9} = \frac{9}{9} = 1$, $.0\dot{9} = \frac{9}{90} = \frac{1}{10}$, $.00\dot{9} = \frac{9}{900} = \frac{1}{100}$

2. If a *Mix'd Circulate* is such that it has no other Figures than what belong to the Repetend, which therefore begins in some Integral Place, it's turn'd into a Vulgar Fraction by making the Denominator as many Places of 9's as the Repetend has Places; and the Numerator is that Repetend with as many o's on the right as there are Integral Places

Places in the given Circulate: So, $\dot{3}.\dot{4} = \frac{340}{99}$; $34.\dot{3}\dot{4} = \frac{3400}{99}$; $34\dot{3}.\dot{4} = \frac{34000}{99}$; and ſo on. The Reaſon is plain; for if in any of theſe Examples you ſuppoſe the Point ſet on the Left of all the Figures, it becomes a *Pure* Circulate, whoſe Finite Value has for its Numerator the Repetend, and as many 9's for its Denominator: Wherefore if the Point is ſet forward where it was at firſt, that removing it forward is in effect multiplying it by 1 with as many o's as the number of Places on the Left of the Point in the given Poſition; ſo that as many o's muſt be ſet on the Right hand of the Repetend, to make the Numerator of the Finite Fraction ſought.

3d. If the Denominator of a Vulgar Fraction conſiſts of 9's, or 9's with o's on the right hand, the Numerator not having more Figures than the 9's in the Denominator, (after equal o's are taken away from Numerator and Denominator) that Fraction reſolves into a *Pure* Circulate, whoſe Repetend is the Numerator, with as many o's on the left as the difference of the number of Places in the Numerator, and 9's in the Denominator; betwixt which Repetend and the Point there muſt be ſet as many o's as ſtand after the 9's in the Denominator; ſo that if the Denominator is 9 &c. without o's, the Repetend begins immediately after the Point.

Exa. (1.) $\frac{34}{99} = .\dot{3}\dot{4}$; (2.) $\frac{26}{999} = .\dot{0}2\dot{6}$; (3.) $\frac{32}{9900} = .00\dot{3}\dot{2}$;

(4.) $\frac{567}{9999900} = .00\dot{0}056\dot{7}$; (5.) $\frac{30}{9900} = \frac{3}{990} = .0\dot{0}\dot{3}$;

The Truth of this *Corollary* appears from its being plainly the Reverſe of the *Theorem*; for ſuch Circulates being form'd, as here directed, their Finite Value will, by the *Theorem*, neceſſarily become the ſuppos'd Fraction.

4th. Suppoſe a Vulgar Fraction as in the laſt, but let its Numerator have more Places than the 9's in the Denominator; that Fraction will be a *Mix'd* Circulate: More particularly if the Figures which the Numerator has more (on the right hand) than the number of 9's in the Denominator, be any of them a ſignificant Figure, or other than o, the Circulate muſt be ſought by actual Diviſion: But if theſe Figures be all o's, the Circulate has no other Figures but what belong to the Repetend, which begins in ſome Integral place: And, to find this Circulate, ſuppoſe theſe o's laſt mention'd to be taken away, then it becomes an Example of *Coroll.* 3; by which find its Circulate, and multiply this by 1 with as many o's as were taken away, *i. e.* remove the Point as many places to the right hand.

Exa. (1.) $\frac{340}{99} = \dot{3}.\dot{4}$ (for $\frac{34}{99} = .\dot{3}\dot{4}$; and this multiply'd by 10 is $\dot{3}.\dot{4}$

(2.) $\frac{24000}{999} = \dot{2}4.\dot{0}$ (for $\frac{240}{999} = .\dot{2}4\dot{0}$; wwich multiply'd by 100, gives $\dot{2}4.\dot{0}$) (3.) $\frac{24000}{99} = 24\dot{2}.\dot{4}$

Scholiums.

(1.) If the Numerator of a Vulgar Fraction conſiſt of the ſame Figures compleatly repeated, rhe Denominator having as many 9's as the Figures in the Numerator, that Fraction is the ſame as if it had but one Period of the Figures repeated, for its Numerator, and as many 9's for its Denominator, (with the o's belonging to the given Denominator, if there were any) thus; $\frac{2424}{9999} = \frac{24}{99}$; for by *Cor.* 3. $\frac{2424}{9999} = .\dot{2}42\dot{4} = .\dot{2}\dot{4} = \frac{24}{99}$. Or thus; 24 : 99 : : 2400 : 9900; therefore 24 : 99 : : 2424 : 9999, from the Property of Proportionals. Hence $\frac{24}{99} = \frac{2424}{9999}$. And $\frac{2424}{99990} = \frac{24}{990}$. This is plain from the other.

Wherefore, if such a Fraction occurs, reduce it first to the Case of *Corol.* 3, and by that find the Circulate sought.

(2d.) If a Vulgar Fraction has a Repeating Numerator, the Repetend having as many Places as the 9's in the Denominator; or with some 0's after so many Places in all the Periods of the Repetend: or in them all but the last on the right; that Fraction is Equal to the Sum of two or more others, each of which will turn to a Circulate by the Rules of *Coroll.* 3d and 4th, whose Sum is therefore the Circulate sought. Thus,

$\frac{32\ 32}{99} = \frac{3200}{99} + \frac{32}{99} = 32.\dot{3}\dot{2}, + .\dot{3}\dot{2}$. (Exa. 2.) $\frac{250\ 250}{99} = \frac{250000}{99} + \frac{250}{99} = 2525.\dot{2}\dot{5}, + \dot{2}.\dot{5}$. (3.) $\frac{45\ 45}{990} = \frac{4500}{990} \left(= \frac{450}{99}\right) + \frac{45}{990} = \dot{4}.\dot{5}, + .0\dot{4}\dot{5}$

5th. From this *Theorem* we also learn, that no *Surd* Root can possibly be a *Circulate*; for Surds have no finite assignable Value, as has been demonstrated in its place, but Circulates have: Wherefore *all* Surds are necessarily Infinite Decimals of the *Incertain* Kind.

THEOREM 7.

Every Vulgar (finite) Fraction is reducible either to a Determinate Decimal, or to a *Circulate*.

DEMON. In the reduction of a Vulgar to a Decimal Fraction, after a Decimal Point in the Quote, we set as many 0's, less by 1, as are necessary to make the Numerator equal at least to the Denominator; and then the division begins, by which the Numerator of the Decimal is found; the Operation being continued by setting 0's to the Remainders successively, and at every Step finding a new Figure in the Quote: But now in Division, how great soever the Dividend be, or however many Figures the Quote contains, the Remainder must always be less than the Divisor: Therefore we can never make so many Steps in this Division as the Divisor expresses, till either we find 0 remaining, or two Remainders the same: For otherwise it would follow, that there are as many Numbers less than the Divisor as the Divisor it self expresses; which is manifestly absurd. Now in the reduction of a Vulgar Fraction, if the Division comes to 0 Remainder, the Decimal is plainly determin'd: But if two Remainders in the Work are found equal, then 'tis certain there must be a Circulation; for the Work will go on for ever, as it has done before betwixt these two equal Remainders, because the Figure to be prefix'd in the next Step is 0, which was also prefix'd to that preceding, and must be to all the succeeding.

SCHOLIUMS.

1. If it happens that there is a Remainder equal to the Given Numerator; then it's plain that all the Figures already set in the Quote will continually circulate, and so be the Repetend, having the same number of Places as that Repetend would have, which would be found by carrying on the Division till two Remainders are found equal; for it's plain that this would happen after you have made as many more Steps in the Work as the number of 0's which make the Numerator equal to the Denominator; as the annex'd Example shews.

$\frac{1}{21}$

$\frac{1}{21} = .\dot{0}4761\dot{9}$ $(= .0\dot{4}76190\dot{4})$

Operation.

```
21) 100 (.04761904
     84
     160
     147
      130
      126
        40
        21
        190
        189
         100
          84
          16
```

Again; If a Remainder occurs equal to the Numerator with any o's on the right, then also you have already the Repetend, which is all the Figures set in the Quote after the Point, excluding as many o's next the Point as are in number equal to these o's on the right of this Remainder. The Reason is manifest.

2d. If an Improper Fraction is given, it will also resolve into an Improper Decimal, either Determinate or Circulate; the fractional part being the Resolution of the fractional part of the given Improper Fraction, and the integral part the same in both.

COROLL. The Repetend in any Circulate can never have more Places of Figures than the Number express'd by the Denominator, less by 1, of that Vulgar Fraction in its least Terms, which is equal in value to the Circulate; *i. e.* which, being reduced, will turn into it: But it may have fewer, as one Example shews, thus; $\frac{7}{13} = .\dot{5}3846\dot{1}$; And this is limited to a Vulgar Fraction in its least Terms, because it's plain that the same Fraction, in whatever Terms, being the same or equivalent Quote, must reduce into the very same Decimal, and consequently if it's a Circulate, the Repetend must be limited by the Denominator of that Fraction in its least Terms.

THEOREM 8.

Part I. If the Denominator of a Vulgar Fraction, in its lowest Terms, has in its composition no Primes but 2 or 5, that Fraction will reduce into a Determinate Decimal, whose Denominator is 1 with as many o's as are express'd by the Index of the highest Power of 2 or 5 (whichever of them has the highest) in the composition of the given Denominator.

Exa. $\frac{3}{40} = .075$, whose Denominator is 1000; and $40 = 2 \times 2 \times 2 \times 5$; so that 2 has the highest Power in the composition of 40, its Index being 3, the number of o's in 1000.

DEMON. Let the Vulgar Fraction be $\frac{N}{D}$, then because D has in it no Prime but 2 and 5, it may be thus represented; $D = 2^n \times 5^m$ (n, m being either equal or different) so that the Fraction is $\frac{N}{2^n \times 5^m}$. Now suppose, according to the Rule of reducing a Vulgar to a Decimal, that N is multiply'd by some Decimal Denominator, or Power of 10, thus, $N \times 10^r$; if this Index r is less than n or m, then $2^n \times 5^m$ cannot measure $N \times 10^r$ (*Theor.* 10, *Ch.* 1.); But if r is equal to n or m, whichever of them is the greater, Then $2^n\, 5^m$ must measure $N \times 10^r$; for $10^r = 2^r \times 5^r$; wherefore 10^r is the Denominator of a Decimal equal to $\frac{N}{D}$.

Part II. If the Denominator D of a Vulgar Fraction $\frac{N}{D}$, in its least Terms, is any other Prime, or has in its composition any other Prime than 2 or 5, (tho' it has these also) that Fraction must resolve into a Circulate. And, the number of o's necessary to finish the Reduction, and discover the first Period of the Repetend, is equal to the number of Places in the Denominator of the finite Value of that Circulate, taken in the Expression of *Theor.* 6.

Exa. $\frac{2}{3} = .66$ &c. or $.\dot{6}$; $\frac{5}{13} = .\dot{3}8461\dot{5}$

DEMON. The Denominator has in it no Prime which is in the composition of the Numerator (because the Fraction is in lowest Terms) and it has some Prime other than 2 or 5, which is therefore in no Power of 10, wherefore it cannot measure the Product of that Numerator by any Power of 10 [*Theor.* 10, *Ch.* 1.] and so cannot make a Determinate Decimal, consequently must turn into a Circulate [*Theor.* 7.]

Again; $\frac{N}{D} = \frac{B}{9 \text{ \&c.}}$ or $\frac{B}{9 \text{ \&c.} \times 10^a}$ by *Theor.* 6, *Part* 2d; in which 9 &c. hath as many Places as the Repetend of the Circulate, and 10^a as many as (i. e. a is equal to) the number of Places betwixt the Point and Repetend: Wherefore 'tis plain that the Places of this Denominator 9 &c. $\times 10^a$ are precisely as many as the o's used in order to finish the first Repetend; Because for every such o there is some Figure placed in the Quote after the Point.

SCHOL. If a Fraction is not in its least Terms, and the Denominator have in it Primes neither 2 nor 5; yet if the same Primes, in the same or a greater Power, be also in the Numerator; Then, because these Primes are out of the Denominator when the Fraction is reduced to least Terms, the Fraction becomes a finite Decimal. Also if any Prime in the Denominator, not 2 nor 5, is not, or is in a lower degree, in the Numerator, that Fraction becomes a Circulate; because, being in lowest Terms, such Primes are all out of the Numerator.

LEMMA.

Let D be any Number in whose composition there is neither 2 nor 5; I say, there is some Number express'd by 9's, as, 9, 99, 999, &c. which is a Multiple of D; *i. e.* take the least Number of 9's, which, written one after another, makes a Number not less than D (which will necessarily consist of as many Places of 9's as there are Places in D); divide that by D, and to the remainder prefix 9, and then again divide; go so on, to every remainder prefixing 9, and dividing by D, there will at last be no remainder; so that the 9's used, written one after another, is a Multiple of D.

Exa. $3 \times 7 = 21$, and $999999 \div 21 = 47619$

DEMON. $\frac{1}{D}$ is a Fraction in its least Terms; and D having neither 2 nor 5 in its composition, this Fraction must reduce to a Circulate [*Theor.* 8.] whose finite value, according to *Theor.* 6, is $\frac{B}{9 \text{ \&c.}}$ or $\frac{B}{9 \text{ \&c.} \times 10^a}$ wherefore these are $::$ *l*, $1 : B :: D : 9$ &c. or 9 &c. $\times 10^a$; but 1 measures B, therefore D must measure 9 &c. or 9 &c. $\times 10^a$. If it's the first Case, the thing propos'd is prov'd; if it's the other, then, because 10^a has no Primes but 2, 5, D, which has neither 2 nor 5, is Prime to 10^a, and consequently it must measure 9 &c. [*Theor.* 6, *Ch.* 1.]

COROL-

COROLLARIES.

1st. If we take a Number of 9's written successively one after another, and whose Number is a Multiple of the least Number of that kind which D measures, D will also measure that assumed Number.

Exa. If D measure 99, it will also measure 9999 or 999999.

2d. If a Number D has neither 2 nor 5 in its composition, then there is some Number express'd by a Number of 3's written successively one after another, as, 3, 33, 333, *&c.* and also a Number express'd by 1's, as, 1, 11, 111, *&c.* which is a Multiple of D; For either D has in its composition some Power of 3, or not: If it has not, then take 99 &c. $= 9 \times 11$ &c. a Multiple of D; and because D, 9, are Prime to one another, but D measures 99 &c. $= 9 \times 11$ &c. consequently D measures 11 &c. And hence again it must measure 33 &c. $= 3 \times 11$ &c. Again; If D has any Power of 3, let $D = 3^n \times a$; And if we take $3^{n+2} \times a$, this measures some Number 99 &c. $= 3^2 \times 11$ &c. And dividing them equally by 3^2, it follows that $3^n \times a$ measures 11 &c. and therefore also it must measure 3×11 &c. $= 33$ &c.

THEOREM 9.

If a proper Fraction in its least Terms has a Denominator which is not measurable by 2 nor 5, it will reduce into a *Pure* Circulate whose Repetend begins immediately after the Point, and has as many Places as the least Number of 9's, which written one after another is a Multiple of the Denominator; which is also the number of 0's necessary to be used, in order to find or bring out the first Period of the Repetend.

Exa. $\frac{5}{21} = .\dot{2}3809\dot{5}$; and 999999 is the least Number of 9's which is a Multiple of 21.

DEMON. $\frac{N}{D}$ being the given Vulgar Fraction, D measures some Number 99 &c. (*per* Lem.) And, supposing 99 &c. the least Number of this kind that D measures, take these : : l, D : 99 &c. : : N : R; then since D measures 99 &c. N must measure R, therefore R is an Integer; for else $N \times q$ (q being an Integer, the Quote of $R \div N$) would be a *Mix'd* Number, *i. e.* the Product of two Integers, a *Mix'd* Number, which is impossible. Again; $D : N : : 99$ &c. $: R$, and $\frac{N}{D} = \frac{R}{99\,\&c.}$ But by *Corol.* 3, *Theor.* 6, $\frac{R}{99\,\&c.}$ is equal to a *Pure* Circulate whose Repetend begins immediately after the Point, and is R with as many 0's on the left as the difference of the number of Places in R and 99 &c, so that it has as many Places as this 99 &c. whence the number of 0's necessary in the reduction to finish the first Period of the Repetend is plainly equal to that number of 9's. Nor can this Repetend possibly contain in it a lesser Repetend beginning immediately after the Point; for supposing that lesser Repetend to be A, and the Denominator 9 &c. then is $\frac{N}{D} = \frac{R}{99\,\&c.} = \frac{A}{9\,\&c.}$ (See *Schol.* 1, *Theor.* 6.) and N, D being incommensurable, D measures 9 &c, which having fewer 9's than the other 99 &c. this other is not the least which D measures, contrary to Supposition: Wherefore 99 &c. the least number of 9's that is a Multiple of D, is the number of Places in the least Repetend of a *Pure* Circulate to which $\frac{N}{D}$ resolves; which is also the least number of 0's necessary in the reduction to finish the first Period of the Repetend.

SCHOL.

SCHOL. If an Improper Vulgar Fraction is given, take out the Integral Part, and the *Theorem* applies to the remaining Fraction: Or that Improper Vulgar becomes an Improper Decimal or Mixt Circulate, whose Fractional Part, taken by it self, is a Pure Circulate: So that every Vulgar Fraction (proper or improper) whose Denominator is not measurable by 2 nor 5, becomes a Circulate whose Repetend begins immediately after the Point.

THEOREM 10.

If the Denominator of a Vulgar Fraction $\frac{N}{D}$, in its lowest Terms, has in its composition any Powers of 2 or 5 with other Primes, take out all these Powers of 2 and 5, and take the Result or Product of all the remaining Primes, *i.e.* divide D by 2 and 5 as oft as possible without a Remainder, and mark the last Quote; the given Fraction will reduce to a Circulate, *Pure* or *Mixt*, whose Repetend has as many Places as the least number of 9's, which is a Multiple of that last Quote; and it begins after so many Decimal Places as are express'd by the Index of the highest Power of 2 or 5, whichever of them has the highest Power involv'd in D.

Exa. $\frac{13}{420} = .03\dot{0}95238\dot{}$; and $420 = 2 \times 2 \times 5 \times 3 \times 7$; out of which all the 2's and 5's being taken, there remains $3 \times 7 = 21$, and the least number of 9's, which is a Multiple of this, is 999999

DEMON. $\frac{N}{D}$ resolves into a Circulate, because D has in it some other Prime than 2 or 5, [*Theor.* 8.] but again D being suppos'd to have in it some Power of 2 or 5, or of both, may be represented thus; $D = A \times 2^n$, or $A \times 5^m$, or $A \times 2^n \times 5^m$ [which last may represent all the Cases; for if the Index n or m is o, that Factor is expung'd] so that A is a Number which has in it no power of 2 or 5; Then is $\frac{N}{D} = \frac{N}{A \times 2^n \times 5^m} = \frac{N}{A} \times \frac{1}{2^n \times 5^m}$; but $\frac{1}{2^n \times 5^m}$ being a Fraction whose Denominator has no Prime but 2 or 5, it is (by *Theor.* 8.) resolvable into a Determinate Decimal whose Denominator is 10^r (r being $= n$ or m, which soever of them is the greater) and may therefore be express'd thus, $\frac{a}{10^r}$: Hence $\frac{N}{D} = \frac{N}{A} \times \frac{a}{10^r} = \frac{Na}{A} \div 10^r$. Again; $\frac{Na}{A}$ may be an Improper Fraction, but cannot be an Integer; for, if it is, then $\frac{Na}{A} \div 10^r \left(= \frac{N}{D}\right)$ is necessarily an Integer, or a determinate Decimal; either of which is impossible, because $\frac{N}{D}$ is suppos'd to be such that it resolves into a Circulate: Wherefore A being a Number not measurable either by 2 or 5, $\frac{Na}{A}$, if it's a proper Fraction, resolves into a *Pure* Circulate, whose Repetend begins immediately after the Point (by *Theor.* 9). And if it's an improper Fraction, it resolves into a *Mixt* Circulate, whose Finite Part is an Integer, and the other Part a *Pure* Circulate; and therefore the complete Quote is a *Mixt* Circulate whose Repetend begins immediately after the Point. Now this Decimal Quote being divided by 10^r (as $\frac{Na}{A} \div 10^r$ directs) it's plain the Figures of it will not be chang'd; only the Decimal Point will be remov'd as many places to the left hand as there are Units in

in r; the Effect of which is plainly this, that whereas the Repetend began immediately after the Point, it begins now after so many places as r expresses; which is the thing to be shewn. *Observe* also, that as $\frac{N\,a}{A}$ makes a *Pure* or *Mixt* Circulate, so does $\frac{N}{D}$; the reason of which is manifest; for in the first Case the Point is remov'd by setting only 0's on the left hand of the Repetend; and in the other Case the Integral part supplys all or some of those places of 0's before the Repetend, and consequently makes a *Mixt* Circulate.

PROBLEM I.

To add Circulates.

RULE. Make them all *Similar*, (by *Theor.* 5.) then take the Sum of the Repetends upon a separate Paper, and divide it by a Number consisting all of 9's, as many as the number of Places in the Repetend; the remainder of the division is the Repetend of the Sum, to be set under the Figures added, with 0's on the left hand if it has not as many Places as the Repetends: The Quote is to be carried to the next Column, and the rest of the Addition done by the common Rules.

Exam. 1.	*Ex.* 2.	*Ex.* 2. reduced.	*Ex.* 3.	*Ex.* 3. reduced.
$.45\dot{3}$	$3.0\dot{4}$	$3.04\dot{4}\dot{4}$	$67.3\dot{4}\dot{5}$	$67.345\dot{4}5454\dot{5}$
$.06\dot{8}$	$8.45\dot{6}$	6.4566	$8.\dot{6}2\dot{1}$	9.621621621
$.32\dot{7}$	$23.\dot{3}\dot{8}$	23.3738	$.\dot{2}\dot{4}$	$.242424242$
$.94\dot{6}$	$.2\dot{4}\dot{8}$	$.2484$	$.\dot{8}$	$.888888888$
$1.79\dot{6}$		$33.12\dot{3}\dot{4}$		$78.098\dot{3}8929\dot{8}$

Ex. 4.	*Ex.* 4. reduced.	*Ex.* 5.	*Ex.* 5. reduced.
267.3456	$267.3456\dot{0}\dot{0}$	$.47836$	$.47836\dot{0}0000\dot{0}$
$33.\dot{8}$	39.888888	$.8\dot{7}2\dot{5}$	$.87257257257$
$.6\dot{7}\dot{2}$	$.672727$	$.\dot{3}\dot{9}$	$.39393939393$
	$307.9072\dot{1}\dot{6}$		$1.74487\dot{1}9665\dot{1}$

Explanation of the Examples.

In *Exa.* 1. the Repetends are all upon one Figure in the same place, and their Sum is 24, which contains two 9's, and 6 remaining; and so 6 is set in that Sum as the Repetend, and 2 carried to the next Column. In *Ex.* 2d reduced to Similar Circulates whose Repetends have two Figures, the Sum of these Repetends is 232, which divided by 99, the Quote is 2, and 34 remains, so that 34 is the Repetend of the Sum, and 2 is carried to the next Column. In *Ex.* 3d reduced to Similars, the Sum of the Repetends is 2389296, which divided by 999999, the Quote is 2, and 389298 remains, which is the Repetend of the Sum, and 2 carried to the next Column. In *Ex.* 4th & 5th there is a finite Decimal, which is also reduced to the form of a Circulate by 0's annex'd to it, which, *observe*, are set down only for Form sake, since they do not alter the Sum. And in such Examples the

the Similar Repetends will always begin after the last Figure of that Finite Decimal which has the greatest number of Decimal Places.

Observe also, That if the Repetend of the Sum consists of the same Figure repeated, the true Repetend is but that one Figure; as in the following Example, the Sum, according to the preceding Rule, comes out $.8\dot{2}2$, which is the same thing as $.8\dot{2}$

Example.

$$\begin{array}{r} .3\dot{4}\dot{6} \\ \underline{.4\dot{7}\dot{5}} \\ .8\dot{2}\dot{2} = .8\dot{2} \end{array}$$

DEMON. By *Theorem* 6, the Finite Value of a *Pure* Circulate is a Fraction whose Numerator is the Repetend, and its Denominator a Number of as many Places of 9's, with as many 0's on the right as there are 0's betwixt the Point and Repetend. Now let the Similar Repetends of several Circulates be added, their Sum is a Numerator to the common Denominator, and this Fraction is the Value of the Sum of these Circulates. Call the Numerator, or Sum of the Repetends, s, and the Sum sought is $\frac{s}{99\,\&c. \times 10\,\&c.} = \frac{s}{99\,\&c.}$ of $\frac{1}{10\,\&c.}$, so that the Sum sought is the Fraction $\frac{s}{99\,\&c.}$ refer'd to an Unit of the Value of the Place next the Repetend on the left hand: Consequently as oft as the Denominator 99 &c. is contain'd in the Numerator s, that Fraction is equal to so many Units of the Value of that next place; and the Remainder is the Numerator of a Fraction having the same common Denominator 99 &c. and to be refer'd to an Unit of the same Place or Value: Wherefore 'tis evident that the remainder of the Division (of $s \div 99$ &c) being placed as a Repetend in the same places as the Repetend added (supplying what places it wants with 0's on the left hand) and the Quote being carried to the next place, the rest of the places added in common form, we have the true Sum sought in all Cases, whether of *Pure* or *Mix'd Circulates.*

PROBLEM 2.

To subtract Circulates.

RULE. Make the Subtrahend and Subtractor *Similar Circulates,* and subtract as they were Finite Decimals: Then, if the Repetend of the Subtractor is a lesser Number than that of the Subtrahend, the Figures in the remainder that stand under the given Repetends (*i. e.* that are their Difference) is the Repetend of the Difference sought; But if the Repetend of the Subtractor is greatest, subtract 1 from the Repetend of the remainder, and the Figures that stand, after this 1 is subtracted, under the given Repetends, make the Repetend of the difference.

Exa. 1.	*Ex.* 2.	*Ex.* 3.	*Ex.* 4.
$8.46\dot{7}$	$24.3\dot{8}\dot{4}$	$.4\dot{2}\dot{7}$	$4.\dot{3}\dot{7}$
$.7\dot{3}\dot{5}$	$9.0\dot{7}\dot{2}$	$.0\dot{3}\dot{4}$	$.\dot{1}\dot{7}$
$7.7\dot{3}\dot{2}$	$15.3\dot{1}\dot{2}$	$.3\dot{9}\dot{2}$	$4.\dot{2}\dot{0}$

Ex. 5.

Ex. 5.	*Ex.* 6.	*Ex.* 7.	*Ex.* 8.
$3.5\dot{3}\dot{6}$	$.7\dot{4}\dot{2}$	$3.85\dot{6}\dot{4}$	$.46\dot{3}\dot{7}$
$2.4\dot{1}\dot{4}$	$.4\dot{1}\dot{8}$	$.03\dot{8}\dot{2}$	$.37\dot{3}\dot{7}$
$1.1\dot{2}\dot{2}$	$.3\dot{2}\dot{4}$	$3.81\dot{8}\dot{1}$	$.09\dot{0}\dot{0}$
$1.1\dot{2}$		or $3.\dot{8}\dot{1}$	$.09$

These *Examples* are so easily compared with the Rule, I shall not insist upon it.

DEMON. If two pure Circulates are Similar, and if the lesser is to be subtracted from the greater, the Reason of the Rule is manifest. But in mixt Circulates the Repetend of the Subtractor may be greater than that of the Subtrahend; and in this Case to follow common Rules, we should add the common Denominator of the Finite Value of the Circulating Parts, with relation to an Unit of the place next the Repetend, the common Denominator is 99, &c. consisting of as many 9's as the places of the Repetend (as has been already explained;) But by Subtracting in the common way, 'tis plain we do add 10, &c. the 0's being as many as the Places of the Repetend; now it's evident, that if instead of 99, &c. we add 100, &c. (this having as many 0's as the other has 9's) we have added 1 too much; and therefore 1 is to be taken from the remainder according to the Rule; the rest is obvious. So in *Ex.* 3. the Repetend having two Places, the Finite Values of the Circulate Parts are $\frac{27}{99}$ of $.1$ and $\frac{34}{99}$ of $.1$; but 34 cannot be taken from 27, therefore I Subtract in common Form, whereby I do add 100 to the Repetend 27, which makes the remainder 93: But because I should only have added 99, I take one from the remainder and it is 92; then because in Subtracting the Circulate Parts, 1 was borrowed from the next Place (for we considered the Circulate Parts as Fractions referred to an Unit of that next Place) therefore 1 is added to the next Place of the Subtracter, and so the Work is carried on.

PROBLEM 3.

To Multiply Circulates.

RULE. Express Circulates by their Finite Values, and then Multiply by the Rule of Vulgar Fractions, reducing and compleating the Answer as the Question requires: And particularly, carry on the Division of the Product of the Numerators by that of the Denominators till 0 remain; or till you come at a Repetend. But if this does not soon happen, then it may be left off at any place you please: But if you are content to have the Product in a Vulgar Fraction, you have it already compleatly in the Product of the two Finite Values of the given Numbers.

The *Reason* of this Rule is manifest; because it's reduced to that of Vulgar Fractions, which is demonstrated in its place; see the following Examples.

Exam. 1. To Multiply $8.4\dot{7}$ by .68, having reduced the first to its Finite Value it is $\frac{763}{90}$, which Multiplied by .68 or $\frac{68}{100}$ produces $\frac{51884}{9000}$ which being reduced to a Decimal is $5.764\dot{8}$ as in the Margin. For the Divisor being

763	9000) 51.884	
68	5.7648	
0124		
4578		
51884		

9000, I first take off three places from the Dividend, which is dividing it by 1000, then I divide this by 9, which gives for a quote 5.764 and 8 remains, to which 0 being prefixt, the next Figure in the quote is 8, and 8 again remains, therefore 8 is repeated.

Exam. 2. To Multiply $7.68\dot{4}$ by $.4\dot{5}$. Being reduced to their Finite Values they are $\frac{6916}{900}$ and $\frac{41}{90}$ whose Product is $\frac{283556.}{90 \times 900,}$ which being reduced to a Decimal is, $3.50069135802\dot{4}$ All that needs be said as to the Operation is, that in the Division of $31.506\dot{2}$ by 9, when we have got the quote so far as 3.5006 then to every succeeding Remainder the Repetend 2 is prefixt; and so the Work is carried on, till there is a Circulation as marked in the Example.

6916	9	283.556
41	9	31.5062
6916		3.500691358O2
27664		
283556		

Exam. 3. To Multiply $65.7\dot{2}\dot{3}$ by 4.6; the first reduced to its Finite Value is $\frac{65066}{990}$ which Multiplied by 4.6 or $\frac{46}{10}$ produces $\frac{2993036}{990 \times 10}$ equal to this Decimal $302.32\dot{6}\dot{8}$. which is found thus: for the two 0's in the Denominator 990×10 I take off two Places in the Numerator, and then divide by 99; which gives for a quote 302.32, and 68 remains, which being $\frac{68}{99}$ of an Unit of the Value of the last place of the quote, is therefore to be placed after it as a Repetend.

65066	99	29930.36
46		2 2 6 8
390396		$302.32\dot{6}\dot{8}$
260264		
2993036		

Observe; the dividing by 99 is here done by *Case* 5. § 2. *Chap* 6. *Book* 1. with this difference only, that I have here placed the Figures of the Operation, and also the quote below the Dividend; and every quote Figure under the first, and not the last Figure of the Dividend, as is there done.

Exam. 4. To Multiply $74.0\dot{3}6\dot{7}$ by $4.7\dot{5}$; their Finite Values are $\frac{73624}{9900}$ and $\frac{428}{90}$ whose Product is $\frac{31511072}{9900 \times 90}$ which reduced is $35.365\dot{9}618406285072951\dot{7}\dot{3}$

For having taken three Places from the Product of the Numerators for the three 0's in the Denominator; I proceed to divide first by 9 which quotes $3501.230\dot{2}$ and then I divide by 99, which gives the quote mentioned; for it's plain, that having brought the Division so far as is here set down, the last remainder is 95, the same with a preceeding remainder belonging to the fifth step of the Division: Wherefore all the Figures in the quote, from the sixth which is 9, do Circulate.

$73624 \times 428 = 31511072$
then
9 | 31511.072.
99 | 3501.2302222222222222222222
  3659518300 8407 9417395
    6 4 5 5
35.36596184062850729517 3

SCHOL.

SCHOLIUM. As this *Rule* is univerſal, ſo it is eaſily kept in mind, if you but remember the Rule for finding the Finite Value of Circulates; nor is it much more tedious than the Multiplication of Finite Decimals, conſidering how eaſily the Finite Value of a Circulate is found; and how eaſy it is to divide by their Denominators, which conſiſt all of 9's, or with 0's; as the preceding Examples ſhew.

There are other Rules for this Multiplication, in ſome things different from the general Rule, but little or nothing eaſier or ſhorter in the Operation; and therefore I might reaſonably paſs them over. Yet that you may know the different ways of managing Circulates, and chuſe as you like beſt, I ſhall here alſo explain the Multiplication of Circulates in two particular Caſes, in order to which; mind that we call that given Number the Multiplier which has feweſt ſignificant Figures.

Caſe 1. The Multiplicand being a Circulate, and the Multiplier an Integer or Finite Decimal, *Simple* or *Mixt*.

RULE. Multiply by each Figure of the Multiplier, Thus; take firſt the Product of that Repetend (of the Multiplicand) and divide it by a Number conſiſting all of 9's, as many as the Number of Places of the Repetend: Write down the remainder in the Product, and carry the quote to the Product of the next Place, and go on with the other Places in common Form: And obſerve that this remainder is a Repetend in every partial Product, and if it has not as many Places as the Diviſor, or Repetend of the Multiplicand, you muſt ſupply the Defect with 0's on the left; and in this State ſet it in the Product as the Repetend. When you have thus got all the partial Products for every Figure of the Multiplier; make all the Repetends ſimilar, which is done by drawing them all out as far as the firſt; then add them by *Probl.* 1. the Sum is the Product ſought, in which ſet the Decimal Point according to the common Rule.

Ex. 1.

$$\begin{array}{r} 8.\dot{4}7 \\ .68 \\ \hline 678\dot{2} \\ 508\dot{6}6 \\ \hline 5.76\dot{4}8 \end{array}$$

Ex. 2.

$$\begin{array}{r} 65.7\dot{2}\dot{3} \\ 4.6 \\ \hline 3643\dot{3}\dot{9} \\ 26289\dot{2}\dot{9} \\ \hline 302.32\dot{6}\dot{8} \end{array}$$

In this *Ex.* 2d. $23 \times 6 = 138$ which divided by 99, quotes 1 and 39 remains; therefore 39 is the Repetend of the Product, and 1 is carried to the next Place, or to the Product 6×7, and ſo that Line is carried on.

Again $23 \times 4 = 92$ which divided by 99, the quote is 0, and 92 remains; which is therefore the Repetend of the Product, the reſt of which is found by the common Rule. But to make this ſimilar to the other, it's reduced to 29; then in ſumming the two partial Products, $39 + 29 = 68$ being leſs than 99, is the Repetend of the Sum, and 0 carried to the next Column.

Ex. 3.

$$\begin{array}{r} 52.\dot{6}7\dot{8} \\ .43 \\ \hline 1583\dot{0}3\dot{6} \\ 2110\dot{7}14\dot{7} \\ \hline 22.690\dot{1}8\dot{3} \end{array}$$

Here $678 \times 3 = 2034$; and this divided by 999 quotes 2, and 36 remains, therefore the Repetend of the Product is $\dot{0}3\dot{6}$, and the quote 2 is carried to the next Place. Again, $678 \times 4 = 2712$ which divided by 999 quotes 2, and 714 remains, which is the Repetend of the Product: The reſt of the Work is obvious.

Case 2. The Multiplier being a Circulate, whatever the Multiplicand is. *Rule.* Take the Finite Value of the Multiplier, and by its Numerator multiply the Multiplicand, by the Method of *Case* 1. then divide the Product by the Denominator.

```
Exa.  7.684̇     90) 315.062̇
        41                3.5006̇691358024̇
      ─────
       7684̇
     307377̇
     ──────
    315.062̇
    ───────
```

To Multiply $7.68\dot{4}$ by $.4\dot{5} = \frac{41}{90}$, the Product is $3.500\dot{6}691358024\dot{4}$ compare this with the *Ex.* 2. to the general Rule, it is the same Example, and the Answer the same. And the Reason and Method of Operation being also obvious, I shall insist no more upon it.

Problem 4.

To divide Circulates.

Rule. Express Circulates by their Finite Values, and then apply the Rule of Vulgar Fractions.

Example. To divide $22.690\dot{1}8\dot{3}$ by $52.7\dot{6}7\dot{8}$; being reduced to their Finite Values they are $\frac{22667493}{999000}$ and $\frac{527151}{9990}$; and the first being divided by the other quotes $\frac{22667493}{52715100}$ (for the two Denominators having 9990 as a common Factor, the quote is reduced to this) whose Value in a Decimal is .43.

Scholium. As Multiplication was explained in two particular Cases differing from the general Rule, so may Division; thus,

Case 1. The Dividend being Circulate, but not the Divisor; *Rule.* Divide as they were Finite Decimals, carrying on the Operation by applying the Repetend so oft till either the Quote circulate; or till you have a sufficient Number of Places: But because in many Cases the Circulation of the Quote will not happen till after a very long Operation, if you would have a compleat Quote, you must take it by the preceding general Rule in a Vulgar Fraction.

```
Exa.                          Exa.
.68) 5.7648̇ (8.47             8) 46.5287 (5.816091
     5 44                        40
     ────                        ──
       324                        65
       272                        64
       ───                        ──
        528                        12
        476                         8
        ───                        ──
         52                         48
                                    58
                                    ────
                                    0072
                                      72
                                     ───
                                     008
                                       8
                                      ──
                                       8
                                      ──
                                       0
```

In

In *Exa.* 1. the Repetend is 8; For the Quote being brought to 847, the Remainder is 52, which is the same as the preceding Remainder; to which the same Repetend 8 being prefix'd, the same Quote 7 must continually come out.

In *Exa.* 2d, after one Period of the Repetend is employ'd, we come at this Quote 58160; but to come at a Circulation in the Quote, we must proceed two Steps farther, by employing again the same Figures of the Repetend in order; and after the two first, *viz.* 2 and 8, are used a second time, we have the same Remainder 0 which was upon the third preceding Step, and therefore the same Figures will repeat again, and so the Quote is 5.816091

Case 2d. If the Divisor is a Circulate, whatever the Dividend is; take the Finite Value of the Divisor; and by its Denominator multiply the Dividend, by *Probl.* 3d, (either by the general Rule or the particular Case 1); then divide the Product by the Numerator (according to the preceding Case, if that is Circulate) and you have the Quote sought.

Exa. 1. To divide $5.764\dot{8}$ by $8.4\dot{7} = \frac{763}{90}$. First I multiply $5.764\dot{8}$ by 90, the Product is 51.884, which divided by 763 quotes .68.

Exa. 2. To divide $\dot{3}.500\dot{6}91358024\dot{4}$ by $7.68\dot{4} = \frac{6916}{900}$, I multiply $\dot{3}.500\dot{6}91358024\dot{4}$ by 900, the Product is $3150.6\dot{2}$, which divided by 6916, quotes $.4\dot{5}$

CHAP. V.

Of Logarithms.

DEFINITION.

LOgarithms are Numbers so contriv'd and adapted to other Numbers that the Sums and Differences of the former correspond to, and shew, the Products and Quotes of the other, and also their Powers and Roots.

SCHOL. 1. This Definition expresses in general the Design and Use of the Numbers call'd *Logarithms*; but, for the more strict and etymological Sense of the Word *Logarithm*, and other Definitions deduced from it more immediately, they will be better understood after we have explain'd the Foundation of their Contrivance; which you have in this

LEMMA.

Take any Geometrical-Progression of Numbers beginning with 1, whose second Term call a, the Series is $1 : a^1 : a^2 : a^3 : a^4 :$ &c. $: a^n$, whereof every Term after 1 is some Power of the second Term a, their Indexes being a Series in Arithmetical Progression, which express the Distances of the several Terms after 1.

From the nature of this Geometrical Series, and what has been explain'd in *Book* 3, *Theor.* 6, 7, 8, these Consequences are manifest, *viz.*

COROLLARIES.

1st. The Product or Quote of any two Terms is also a Term of the Series whose Index (or Distance after 1) is the Sum or Difference of the Indexes of these two Terms. *Exam.* $a^3 \times a^4 = a^7$. $a^7 \div a^3 = a^4$. Universally, $a^n \times a^m = a^{n+m}$, and $a^n \div a^m = a^{n-m}$.

2d. Any

2d. Any Power of any of these Terms is a Term of the Series whose Index is the Product of the Index of that Power by the Index or Distance of that Term from 1. *Exam.* The Square of a^3 is a^6; Universally; the n Power of a^m is a^{nm}. *Reversely*; If the Index of any Term is Multiple of any Number, then, being divided by that Number, the Quote is the Index of a Term in the Series, which is such a Root of the Term whose Index is divided as the Divisor denominates. *Exa.* The Cube Root of a^6 is a^2. Universally; the n Root of a^m is $a^{m \div n}$.

3d. If from the double of any Index, or the Sum of any two Indexes, be subtracted the Index of another Term, the Difference is the Index of a Term in the Series which is a third or fourth $::$ l to the 2d or 3d Terms whose Indexes are given. *Exa.* 1st, $a^2 : a^5 :: a^5 : a^8$; where $8 = 5 + 5 - 2$. *Ex.* 2d, $a^2 : a^5 :: a^7 : a^{10}$; where $10 = 5 + 7 - 2$. Universally; $a^n : a^m : a^{2m-n}$ are $\div l$; for by common Rules a third to $a^n : a^m$ is $a^m \times a^m \div a^n$; but $a^m \times a^m = a^{2m}$, and $a^{2m} \div a^n = a^{2m-n}$. Again; $a^n : a^m :: a^r : a^{m+r-n}$; for $a^m \times a^r = a^{m+r}$, and $a^{m+r} \div a^n = a^{m+r-n}$.

SCHOL. 2. Here then we have the *Fundamental* Grounds of the *Invention* of *Logarithms*: For 'tis obvious that the Indexes or Distances of the several Terms of a Geometrical Series from the first Term 1, are Numbers answering to the preceding Definition of *Logarithms*, for those Numbers that make the Geometrical Series; which I shall more particularly explain and apply: But first *Observe*, that from this Foundation is deduced the common Definition of *Logarithms*, viz. *Numbers in Arithmetical Progression answering to others in Geometrical Progression.* Again, *observe*, that in the more strict sense of the Word it signifies *a Number of Ratios*: And, to understand the reason of its application here, consider, That in a Geometrical Series the Ratio of the Extremes is compos'd of as many equal Ratios as the number of Terms less 1, or as the Number expressing the Distance of the one Extreme after the other; which Distance is therefore call'd the *Logarithm* of the Ratio of the one Extreme to the other; and if one Extreme is 1, it's call'd simply the *Logarithm* of that other; but it strictly signifies the Logarithm of the Ratio of 1 to that other.

Let us now suppose a to be any Number, as 2; the Geometrical Series from $1 : 2$ is $1 : 2^1 : 2^2 : 2^3 :$ &c. or $1 : 2 : 4 : 8 : 16 :$ &c. And if this Series is carried to any length, and the several Terms be dispos'd orderly in a Table, and against them be set their Indexes or Distances from 1, (*viz.* their *Logarithms*) setting 0 against 1, because it's not distant from it self, as in the following Table; then from what's explain'd it is evident that we can, by means of these Indexes or Logarithms, find the Product or Quote of any two Terms of the Geometrical Series, without actual Multiplication or Division; also any Power or Rational Root of any Term; and, lastly, a third or fourth Proportional to any two or three of them, supposing the Table carried to a sufficient extent, as in the following Examples; the Rules of which are contain'd in the Consequences to the preceding *Lemma*, and which 'twill be useful to repeat here in somewhat a different form, with a direct Regard to the Practice and Use of Logarithmick Tables, under the Title of

The Fundamental General RULES *for the Use and Practice of* LOGARITHMS.

I. Add the Logarithms of any two Numbers, the Sum is a Logarithm, against which in the Table stands the Product of these two Numbers:

Exa. $8 \times 32 = 256$, whose Logarithm is $8 = 3 + 5$, the Logarithms of 8 & 32.

II. Take the Difference of the Logarithms of any two Numbers, it is that Logarithm against which stands the Quote of the greater divided by the lesser of these two Numbers. *Exa.*

Exa. $2048 \div 128 = 16$, whose Logarithm is $4 = 11 - 7$, the Logarithms of 2048 & 128.

III. Multiply the Logarithm of any Number by any Number, the Product is a Logarithm against which stands that Power of the Number whose Logarithm is multiply'd, denominated by the Multiplier, *viz.* the n Power, if the Multiplier is n.

Exa. $16^3 = 4096$, whose Logarithm is $12 = 4 \times 3$; the Logarithm of 16 (*viz.* 4) multiply'd by 3, the Index of the Power sought.

IV. Divide the Logarithm of any Number by any Number, and if there is no Remainder the Quote is a Logarithm against which stands that Root of the Number whose Logarithm is divided, denominated by the Divisor, *viz.* the n Root of the Divisor is n.

Exa. $4096^{\frac{1}{3}} = 16$, whose Logarithm is $4 = 12 \div 3$, the Logarithm of 4096 (*viz.* 12) divided by the Divisor 3.

V. From the Double of the Logarithm of any Number, or the Sum of the Logarithms of any two Numbers, subtract the Logarithm of any Number, the Difference is a Logarithm against which stands a Number that is a third or fourth Proportional to these two or three given Numbers.

Exa. 1) $4 : 16 : 64$ are $\div\!\!\div l$, whose Logarithms are $2 . 4 . 6$ $(= 4 + 4 - 2.)$
Exa. 2) $8 : 32 :: 256 : 1024$, whose Logarithms are $3 . 5 . 8 . 10$ $(= 5 + 8 - 3.)$

TABLE *whence these* Examples *are taken.*

No.	Log.	No.	Log.	No.	Log.
1	0	64	6	4096	12
2	1	128	7	8192	13
4	2	256	8	16384	14
8	3	512	9	32768	15
16	4	1024	10	65536	16
32	5	2048	11	131072	17

SCHOL. 3. Any Arithmetical Progression beginning with 0 may be apply'd as Logarithms to any Geometrical one from 1: For tho' they will not be Logarithms according to the stricter meaning of the Word, (*i. e.* the number of Ratios from 1) yet they will answer the other Definitions, and the preceding Fundamental Rules: Which will be manifest from the common Properties of Arithmetical and Geometrical Progressions: For any three or more Terms, taken in either Series, are continuedly proportional in their kind; and any four Terms are proportional, whereof the first and second are as far distant as the third and fourth.

Geom. $1 : A : B : C : D : E : F :$ &c.
Arithm. $0 . a . b . c . d . e . f .$ &c.

Whence the Correspondence of the Arithmetical, as Logarithms to the Geometrical, according to the preceding Rules, is plain. Thus, $B \times C = E$ for $1 : B :: C : E$. Also $b + c = e$ for $0 . b : c . e$. Again, $F \div D = B$ for $1 : B :: D : F$. Also $f - d = b$ for $0 . b : d . f$.

The Rule for Powers is deduced thus; $B^2 = D$ for $1 : B :: B : D$; also $2b = d$ for $0 . b : b . d$. Again, $B^3 = F$, for $1 : B : D : F$ are $\div\!\!\div l$; then $3 \times b = f$, for $0 . b . d . f$ are $\div\!\!\div l$. And so it goes on thro' all the rest of the Powers. And the Rule for Roots is but the Reverse of this.

For the finding a 3d or 4th Proportional, the Rule must also still be good, because the Rules for Multiplication and Division are good.

SCHOL.

SCHOL. 4. Any Arithmetical Progression whatever may be apply'd as Logarithms to a Series geometrical from 1; but if the Logarithm of 1 be any other than 0, the preceding Rules will not answer, and instead of them we must put these:

1°. From the Sum of the Logarithms of two Numbers take the Logarithm of Unity, the difference is the Logarithm of the Product.

2°. To the difference of the Logarithm of two Numbers add the Logarithm of 1, the Sum is the Logarithm of their Quote.

The *Reason* of these Rules is plain from the Proportionality of the Terms with 1; for 1 is to the Multiplier as the Multiplicand is to the Product; or, 1 and the Multiplier are at the same distance as the Multiplicand and Product; So also the Divisor and Dividend are proportional with, or at the same distance as the Quote and Unity; but the corresponding Terms in the Arithmetical Series are also arithmetically proportional; whence the Rules are clear. So, in the preceding Example, suppose the Logarithm of 1, (*viz.* 0) to represent any Number; then, as $B \times C = E$, so $b + c - 0 = e$; and as $F \div D = B$, so $f - d + 0 = b$.

3°. The Rule for finding a 3d or 4th Proportional is the same in all Suppositions, and the Reason the same.

4°. For finding the Power of any Number, multiply its Logarithm by the Index of the Power, (viz. n) and from the Product take the Product of the Logarithm of 1, multiply'd by $n - 1$; thus $B^2 = D$, whose Logarithm is $d = 2b - 0$ (because $0 \,.\, b : b \,.\, d.$) Again, $B^3 = F$, for $1 : B :: D (= B^2) : F$; then $0 \,.\, b : d \,.\, f$, also $d = 2b - 0$; Whence $0 \,.\, b : 2b - 0 \,.\, f$ and $3b - 2 \times 0 = f$, the Logarithm of F, when 0 is the Logarithm of 1, and b the Logarithm of B. The Reasoning will proceed in this manner for ever. Or, the Demonstration of this Rule may be made *Universally*, thus; Let any Number in the Geometrical Series be A, then $1 : A :: A^n : A^{n+1}$; And if the Logarithm of 1 is 0, and that of A is a, then, according to this Rule, the Logarithm of A^n is $na - \overline{n-1}\,a$. Now if this Rule is good in one Case as the n Power, it's good in the next as the $n + 1$ Power; For $1 : A$ and $A^n : A^{n+1}$ standing at the same distance in the Geometrical Series, so do their corresponding Terms in the other Series: Call l the Term corresponding to A^{n+1}, then is l at the same distance from $na - \overline{n-1a}$ (the suppos'd Logarithm or Term corresponding to A^n) as a is from 0; *that is*, $0 \cdot a : na - \overline{n-1} \,.\, l (= \overline{n+1}\,a - na$, by the common Rules) which is therefore the Logarithm of A^{n+1} according to this Rule. But the Rule is shewn to be true for the *Square* and *Cube*, and consequently 'tis true for all above.

5°. To find the Root of any Number; to the Logarithm of the Power add the Product of the Logarithm of Unity by $n - 1$, and divide the Sum by n; the Quote (being integral) is the Logarithm of the Root. The *Reason* of this is contain'd in the last Rule; For if l is the Logarithm of the Root, and ll that of the Power whose Index is n, then is $ll = nl - n - 10$. Hence $ll + n - 10 = nl$, and $\frac{ll + \overline{n-1} \times 0}{n} = l$, which is the Rule.

SCHOL. 5. A Geometrical Series may not only encrease from Unity, but also decrease, thus, $\frac{1}{8} : \frac{1}{4} : \frac{1}{2} : 1 : 2 : 4 : 8$; Universally, $\frac{1}{a^4} : \frac{1}{a^3} : \frac{1}{a^2} : \frac{1}{a}; a : a^2 : a^3 : a^4$ &c. And to these Terms decreasing from Unity their Distances are also Logarithms, with this Variation only in the Practice, That we must consider them as negative Numbers, or Numbers less than 0, because their distance from 1 is upon the opposite side to that of the Integral Terms: And hence we must apply the *Rules* with a Regard to this, *i. e.* by observing

observing the Rules of the Addition, &c. of negative Numbers, as explain'd in *Book* I.

Suppose $a = 2$, then the Series decreasing is $\frac{1}{2} : \frac{1}{4} : \frac{1}{8}$, &c. which reduced to Decimal Fractions, are .5 : .25 : .125 : 0625, &c. And the whole Series encreasing and decreasing with their Logarithms, stand as in the following Scheme.

And these Logarithms being thus taken positive upon the one side of 0, and negative on the other, make still a Series Arithmetical wheresoever you begin it, and so must answer to the Rules.

Exa. $16 \times .125 = 2.000 = 2$, whose Logarithm is $1 = -3 + 4$ or $4 - 3$, the Sum of the Logarithms of .125 and 16

Ex. 2. $4 \times .0625 = .2500 = .25$, whose Logarithm is $-2 = -4 + 2$, the Sum of the Logarithms of .0625 and 4

But *Observe* that the Logarithms of such a mix'd Series may be made all positive to any Term of the Fractional part, by making 0 the Logarithm of any of the Fractions, and then the Logarithms of all above are positive; or by applying any Series of positive Numbers in Arithmetical Progression as Logarithms; for these will answer according to the Rules explain'd in *Scholium* 4, and for the Reasons there also demonstrated. Thus, Suppose 0 the Logarithm of .0625, then the Logarithms of the rest above it are as in this Scheme: And $16 \times .125 = 2$, whose Logarithm is $5 = 8 + 1 - 4$ according to *Rule* 1, *Schol.* 4; 8, 1, 4 being here the Logarithms of 16, .125, 2

.0625 :	.125 :	.25 :	.5 :	1 :	2 :	4 :	8 :	16
−4 :	−3 :	−2 :	−1 :	0 :	1 :	2 :	3 :	4

SCHOL. 6. Whatever Arithmetical Progression we apply to a Geometrical one, they are Logarithms only to that Series to which we apply them, and answer the Ends propos'd only for these particular Numbers; so that if we have Logarithms adapted only to particular Geometrical Series, they would be of little Use. The great End and Design of Logarithms is, the Ease and Expedition of Calculations, by saving the Laborious Work of *Multiplication*, *Division*, and *Extraction of Roots*: But this End could never be compleatly reach'd, unless Logarithms could be adapted to the whole System of Numbers, 1 . 2 . 3 . 4 . &c. And as here lay the Excellence and Merit of the Contrivance, so also the Difficulty; for the Natural System of Numbers, 1 . 2 . 3 . 4 . &c. being an Arithmetical, and not a Geometrical Series, seems rather fit to be made Logarithms, than to have Logarithms apply'd to it: Yet this Difficulty the Excellent Genius of the Renown'd Author, and Unrival'd Inventor of them, [the Lord *Neper*] conquer'd. And in order to understand his Method of Constructing these Logarithms, *Consider*;

Tho' the Whole Natural System of Numbers, 1 . 2 . 3 . 4 . &c. makes not One Geometrical Series, and cannot by any Means be brought within one such Series of Determinate Numbers, yet they may be brought near to it, within any assignable degree of Approximation; which may be conceiv'd in general, thus: Suppose a Fraction infinitely small represented by x, and a Series Geometrical arising from 1 in the Ratio of 1 to $1+x$, this Series is represented by $1 : \overline{1+x}^1 : \overline{1+x}^2 : \overline{1+x}^3 : \overline{1+x}^4$: &c. some of whose Terms must coincide, infinitely near, with the natural Numbers 2 . 3 . 4 . &c. because among Numbers that arise by infinitely-small Increments [as it is in this Case, since the common Ratio is infinitely near to a Ratio of Equality] some of them must exceed, or come short of, any determinate Number by an infinitely little Excess or Defect: Wherefore if in the places of the Terms of this Series, that do approach infinitely near to any of the natural Numbers, we suppose these natural Numbers themselves, then the Series will be a Geometrical Progression to an exactness, which I call *Indefinite* (and not *Perfect*) because the approximation of its Terms to the natural Numbers can never end, but goes on *in infinitum.*

Rrr Now,

Now, as this Imagin'd Geometrical Series comprehends infinitely near the whole System of Numbers 1 . 2 . 3 : &c. so their Indexes comprehend a complete System of Logarithms for the natural System of Numbers, extended to any length we please; and do answer to all the preceding Definitions and Rules: For, tho' the natural System make not by themselves a Geometrical Series, yet they are conceiv'd as a part of such a Series; and so the Logarithms are the Indexes of their Distances from Unity in that Series; or, more generally, they are the corresponding Numbers of an Arithmetical Series apply'd to that Geometrical one.

But again *observe*, that since we cannot assign an infinitely-little Fraction, therefore in the actual construction of Logarithms we must be content with a determinate degree of Approximation: Whence, according as we take x, so in the Series $1 : \overline{1+x} : \overline{1+x}^2$: &c. the Approximation of its Terms to the natural Numbers will be in different degrees; for the lesser x is, the nearer will the Approximation be; but then the more are the Involutions of $2 + x$ necessary to come within any determinate degree of nearness to any of the natural Numbers.

Thus then we may conceive the absolute Possibility of making Logarithms to the natural System 1 . 2 . 3 : &c. to any determinate degree of exactness, *viz.* by assigning a very small Fraction for x, and actually raising a Series in the Ratio of 1 to $1 + x$, and taking for the natural Numbers such Terms of that Series as are nearest to them, and their Indexes for the Logarithms. But *observe* also, that to construct Logarithms in this manner, to such an extent of Numbers, and degree of Exactness, as would be necessary to make Logarithms of any considerable use, is next to Impossible to us, because of the almost infinite Labour and Time it would require: This, however, is an Introduction for understanding the Method of the *Noble Inventor*, who, as he (no doubt) took the Hint of Logarithms from the Consideration of the Indexes of a Geometrical Series, so, to compleat the Invention, he behov'd to lay before him the Idea of a Geometrical Progression comprehending, infinitely near, all the Terms of the Natural Series; but, that the Labour of constructing these Logarithms might not be insuperable, he went to work another Way: For,

From the Foundation already laid down in the Consideration of an Infinite Progression, this Conclusion was obvious, *viz.* That if the natural Numbers were comprehended in this Series to an infinite or indefinite degree of nearness, there must also be an infinite or indefinite number of Means betwixt any two of the natural Numbers, or such Terms of the Progression in whose places we substitute them: And upon this *Principle* he constructed his Logarithms; the Method of which I shall explain in the following *Problems.*

PROBLEM I.

Betwixt any two given Integral Numbers, an indefinite Number of Geometrical Means being suppos'd, 'tis requir'd to find one of them so approximate that it be within an assign'd difference of a given Number which lies in the natural System betwixt the former two given Numbers.

RULE. Let the given Extremes be call'd A (the lesser) and B (the greater) and the given Mean C; Then, betwixt A, B find one Geometrical Mean; and if A, B admit not a rational Mean, find the Approximate to any number of Decimal Places: Call this Mean D; and, if D is less than C, find a Mean betwixt D and B; but if D is greater than C, find a Mean betwixt D and A: Call this second Mean E; then, as E is lesser or greater than C; find a third Mean betwixt E and the first Mean D, if this is contrarily greater or lesser than C; or betwixt E and the opposite Extreme B or A, if D is of the same quality with E (*i. e.* greater or lesser than C, as E is): Call this third Mean F; and, as it is lesser or greater than C, find a fourth Mean betwixt it and that one of the preceding

Means

Means which is next greater than C; but if all the preceding Means are of the same quality with F, find a Mean betwixt F and the opposite Extreme B or A. In this manner go on finding Geometrical Means approaching till you find one within any Difference you please.

Examp. To find a Mean in the Infinite Series betwixt 1 and 10, which approaches to 9 within a Difference of $\frac{1}{1000000}$

Operation. Betwixt 1 and 10 there is not a Rational Mean (for 10 is not a Square Number, and the Mean is the Square Root of 1×10 or 10) but I find one approximate to 7 Places of Decimals (*viz.* the Number in the Denominator of the Fractional Difference) which is 3.1622777 &c. which being less than 9, betwixt it and 10 I find another approximate Mean, which is 5.6234132 &c. which being also less than 9, I find betwixt it and 10 another, which is 7.4989421 &c. and this being also less than 9, I find betwixt it and 10 another, 8.6596432 &c. which being yet less than 9, I find another Mean betwixt it and 10, *viz.* 9.3057204 &c. which is greater than 9, therefore I find a Mean betwixt it and 8.6596432, (the next lesser Mean) which is 8.9768713 &c. which being less than 9, betwixt it and 9.3057204 &c. (the next greater Mean) I find a Mean 9.1398170 &c. greater than 9, wherefore betwixt it and 8.9768713 &c. (the next lesser) I find a Mean 9.0579777 &c. greater than 9. Going thus on, you'll find at the 25th Step this Mean, 8.9999998 &c. which wants of 9 this Fraction, $\frac{2}{10000000} = \frac{1}{5000000}$; which is less than the propos'd Difference, $\frac{1}{1000000}$

So much of the Operation as is here explain'd you see placed in Order, as it was wrought, in the Example of the following *Problem* 2, which will give a clearer view of it; the rest of the Steps are easily imagin'd by these: But *Observe*, that the Means being all Approximates which requir'd Decimal Places in the extraction, therefore the given Numbers 1, 10 are taken in the Operation with as many 0's as the Approximate Mean ought to have, which has the same Effect in the Operation, since 1 : 10 : : 10000000 : : 100000000; so that a Mean betwixt 1, 10 is the same as betwixt 10000000 : 100000000

DEMON. All that's necessary to be said as to the reason of this Operation is, in short, this; That if at every Step of the preceding Operation we conceive the Series to be fill'd up betwixt the given Extremes, in the Ratio that every new Mean makes with the Terms betwixt which 'tis taken, we can thus carry on the Number of Means *in infinitum*; so that the Means thus found are still a part of the infinite number of Means suppos'd to lie betwixt these Extremes: Wherefore, by assuming any two Numbers of the Natural System, we can thus find Approximates to all the intermediate ones within any assign'd Difference, and such too as shall make a Geometrical Series indefinitely near.

SCHOL. 7. Let us now suppose any Series Geometrical from 1, as, 1 : 2 : 4 : &c. or 1 : 10 : 100 : &c. it is plain how, by the Method of this *Problem*, we can find Mean Terms betwixt each of these, so nearly approximate as to make of the whole one Geometrical Series, to an indefinite degree of exactness; and among which we can find Terms approaching within any Difference of any of the Intermediate Natural Numbers. But *Observe* also, That if the Construction of Logarithms requir'd the finding the Approximates to every one of these Intermediate Numbers, the Labour would still be intolerably great; which is prevented by the Consideration of the Fundamental Principles and Rules of *Logarithms*; as in the next *Problem*.

PROBLEM 2.

To Construct *or Find* LOGARITHMS *to the Natural System of Numbers* 1 . 2 . 3 . 4 . &c. *carried to any Extent.*

SOLUTION.

1. Take the Geometrical Progression, 1 : 10 : 100 : &c. to which apply as Logarithms the Arithmetical Series 0 . 1 . 2 . 3 : &c. *viz.* 0 the Logarithm of 1, 1 the Logarithm of 10, and so on; then

2. For the Logarithms of the Intermediate Numbers, the *General Rule* is this: Find, by the preceding Problem, a Mean betwixt 1 and 10, or 10 and 100, or any two adjacent Terms of the Series betwixt which the Number propos'd lies, so approximate that it be within the propos'd Limit of the Number whose Logarithm is sought; for example, so near that it want not $\frac{1}{1000000}$ [whereby, if it's taken less than that Number, 'twill necessarily have in the Integral part a Number wanting 1 of it, and the Decimal part will have 9 in 6 Places immediately after the Point: But if the Denominators of the limiting Fraction has any other Number of 0's, the approximate Mean must have as many 9's immediately after the Point.] Then, betwixt the Logarithm of 1 and 10 (or other two Terms betwixt which the Number lies) find as many Arithmetical Means in the same order as you found Geometrical Means betwixt 1 and 10; and thus you find gradually the Logarithms answering to each of these Geometrical Means, and consequently the Logarithm of the Mean approximate to the Number propos'd, which we therefore take for its Logarithm.

DEMON. That Logarithms thus found to all the Intermediate Numbers betwixt the Terms of the Series 1 : 10 : 100 : &c. are true, and must answer to the preceding Rules, is clear, because they are found by those very fundamental Principles, whereby the Logarithm of a Geometrical Mean betwixt any two Numbers is necessarily an Arithmetical Mean betwixt the Logarithms of these two Numbers: And as all the Geometrical Means thus found are part of the infinite number of Means suppos'd betwixt any two Terms of this Series, so the Arithmetical Means thus found must likeways be their Correspondents among the infinite number of Arithmetical Means lying betwixt the Logarithms of these two Numbers; and hence the Logarithms are truly found, according to the determin'd degree of approximation.

Observe also in the following Example, that because in halving the Sum of two Logarithms to find the Arithmetical Mean there will be Fractions, Therefore, either to prevent this, or to find them in Decimals, the Logarithm of 10 is made 1.000000 (instead of 1); *that is*, 1 with as many 0's as are in the Denominator of the limiting Fraction within which the approximate Mean was determin'd; whence all the other Logarithms will have as many Decimal Places. And the Logarithms thus found may be consider'd either as altogether *whole* Numbers, or as *mix'd*, for the Effect will be the same; because by considering them as *whole* Numbers, they are only the equal Products of what they are when consider'd the other way; and therefore, either way, they are correspondent Terms of an Infinite Arithmetical Progression adapted to an Infinite Geometrical one; for if they are so when consider'd as *mix'd* Numbers, or when the Logarithms of 1 : 10 : 100, &c. are 0 . 1 . 2, &c. they must be so when consider'd as Integral, or when the Logarithms of 1 : 10 : 100, &c. are 000000 . 1000000 . 2000000, &c. since an Arithmetical Progression, equally multiply'd, continues still Arithmetical.

In

In the following Scheme you see the Order of the Operation, whereby the Logarithm of 9 is found, carried to the 8th Step, which is sufficient to illustrate and make plain the rest of the Work for this or any other Example.

Order of the Operation whereby is found the Logarithm of 9.

		Numbers.	Logarithms.			Numbers.	Logarithms.
Given	*A*	1.0000000	0.0000000		*F*	8.6596432	0.9375000
1st Mean	*C*	3.1622777	0.5000000	5th	*G*	9.3057204	0.9687500
Given	*B*	10.0000000	1.0000000		*B*	10.0000000	1.0000000
	C	3.1622777	0.5000000		*G*	9.3057204	0.9687500
2d	*D*	5.6234132	0.7500000	6th	*H*	8.9768713	0.9531250
	B	10.0000000	1.0000000		*F*	8.6596432	0.9375000
	D	5.6234132	0.7500000		*H*	8.9768713	0.9531250
3d	*E*	7.4989421	0.8750000	7th	*I*	9.1398170	0.9609375
	B	10.0000000	1.0000000		*G*	9.3057204	0.9687500
	E	7.4989421	0.8750000		*I*	9.1398170	0.9609375
4th	*F*	8.6596431	0.9375000	8th	*K*	9.0579777	0.9570312
	B	10.0000000	1.0000000		*H*	8.9768713	0.9531250
		Which carried on, the 25th Mean is				8.9999998	0.9542425

In the same manner may we find the Logarithm of any other Number betwixt 1 and 10, or betwixt 10 and 100, &c. *Observe* also, that having found the Logarithm of any Mean betwixt 1 and 10, &c. we may either use the same Extremes 1, 10 for finding any other Mean and its Logarithm, or we may use any other two Extremes whose Logarithms are already known. Thus, having found the Logarithm of 9, To find that of 8, or 2, or any other, we may use 1, 10, or 1, 9. And if we have the Logarithms of 7 and 5, we may use these for finding that of 6; and so of other Cases: And it will be best to chuse the Extremes as near to one another as we can. The Reason of the Work is still the same, because betwixt any two Terms of the natural System we suppose an indefinite number of Means.

Again Observe, That tho' this *General Rule* is good, yet to find all the intermediate Numbers betwixt 1 and 10, 10 and 100, &c. in this manner would be an endless Labour, which is sav'd by the following

Particular RULES.

1°. Having by the *General Rule* found the Logarithm of any Number, the Logarithms of all the superior Powers of it are found by simply multiplying the Logarithm of that Number into the Series of Indexes of all the superior Powers, *viz.* 2 . 3 . 4 . &c. which produces

produces the Logarithms of its Square, Cube, &c. The Reason of this is contain'd in the Third *General Rule*: So having the Logarithm of 2, we have the Logarithms of its Powers, 4 . 8 . 16 . &c. if we multiply the Logarithm of 2 by 2, 3, 4, &c. successively.

2°. Having found by the General Rule the Logarithm of any Power, divide it by the Index of the Power, and the Rule is the Logarithm of the Root: The Reason is contain'd in the Fourth General Rule. Thus having found the Logarithm of 9, its half is the Logarithm of 3, the square Root of 9.

3°. Having found the Logarithms of any two or more Numbers, (as of 2 and 3) we have the Logarithm of their Product (6) by adding their Logarithms into one Sum; as in the First General Rule.

4°. Having the Logarithm of any Number, and of another which measures it, we have, by the Second General Rule, the Logarithm of the Quote or Number by which the lesser measures the greater: Thus, from the Logarithm of 10 take that of 2, and the remainder is the Logarithm of 5.

From these Rules therefore it's plain how, with much less Labour than applying the General Rule to every Number, we can compleat a System of Logarithms: And to go on regularly with the Application, the natural Method is obviously this, *viz.*

Apply the *General Rule* to Prime Numbers, and then by these find the Logarithms of their Powers and Products, thus; By the Logarithms of 2 and 3 we find that of 6, 12, 24, and all in that Progression: Also of 18, 54, 162, and all in that Progression: And, lastly, of 36, 216, 1296, and all in that Progression: Other Applications are easily conceiv'd by this. *Observe* also, that the Prime Numbers, 2 and 5, do not both require the *General Rule*, because we have the Logarithm of 10 assum'd, and 10 is $= 2 \times 5$. Again; it's sometimes as convenient, or rather better, to find the Logarithm of the Square of a Prime Number by the *General Rule*, and then the Logarithm of the Root by the *Particular Rules*; so we may chuse to find the Logarithm of 9 by the *General Rule*, and then its half is the Logarithm of 3.

SCHOLIUM 8. Upon the *Foundations*, and by the *Rules* now explain'd, were the *Logarithms* first calculated, which we have in our present Tables, (tho' they have been constructed anew by Methods incomparably easier). It's true indeed, that in the *First Logarithms* made by the Lord *Neper*, the Logarithms of 1 : 10 : 100 &c. were other Numbers than 0 . 1 . 2 . &c. but afterwards he chang'd them into these, which were, after his Death, further compleated and carried on upon this Foundation by Mr. *Henry Briggs*, by a Method also somewhat different from *Neper*'s, yet equally laborious.

Where-ever you find Tables of *Logarithms*, you'll find also Directions for the Use of them; And therefore since I refer you to other Books for Tables (and you have Books containing nothing but Tables, as *Ozanam*'s and *Sherwin*'s, which last are the best extant) I shall add nothing to the general Rules already delivered, which do sufficiently shew the Practice and Use of *Logarithms*: For what is more to be said as to the Use of Tables relates only to the different Methods of disposing the Numbers and *Logarithms* in the Tables, which every Book of Tables explains; but still there remains a few Articles to be explained, concerning the finding *Logarithms* for Numbers that are not contained in your Tables, or Numbers corresponding to *Logarithms*; which being the result of an Operation with *Logarithms* found in the Table, are not themselves exactly found in the Table: These things I shall explain in the following Problems; but first I'll dispatch these Observations.

1. That there may be great Variety in the Systems of *Logarithms*, which depends in general upon these two things, *viz.* The fundamental Geometrical progression, whose *Logarithms* we assume; for that may be 1 : 2 : 4 : 8, &c. or 1 : 3 : 9 : 27, &c. or 1 : 10 : 100:

100 : 1000, &c. Then the different Arithmetrical Progressions we may assume for the *Logarithms* of this Geometrical one: For thus *Logarithms* may be varied infinitely; yet they will not be all alike convenient. The consideration of which obliged the Inventer to change his first *Logarithms* into others, whose fundamental Progression is 1 : 10 : 100 : &c. and the *Logarithms* 0 . 1 . 2 . &c. which are those now used.

2. The great Advantage and Conveniency of the *Logarithms* now in Use is this, That the Integral part in every *Logarithm* shews how many Figures after the Place of Units the corresponding Number contains, whence that Number is called the *Index* or *Characteristick* of the *Logarithm*. Thus all the Numbers from 1 to 10 exclusive, consist of one Figure: For the *Logarithm* of 1 being 0, and that of 10 being 1, therefore the *Logarithms* of all the intermediate Numbers must be Decimal Fractions, and so have 0 for the Integral part or Characteristick. Again; the *Logarithm* of 10 being 1, and of 100 being 2; all the intermediate Numbers must have 1 for their Integral part or Index; and so on. The Benefit of this is remarkable in finding the *Logarithms* of Numbers that are in a decuple Progression, having the *Logarithm* of any one of them, (and consequently for Decimal Fractions) because the *Logarithm* of 1 being 0, that of 10 being 1, and that of 100 being 2, and so on; it follows that the *Logarithm* of the Product or Quote of any Number by 10, is had by adding 1 to, or substracting 1 from the *Logarithm* of the given Number *A*; because 1 : 10 :: *A* : 10 × *A*, whose *Logarithm* is therefore the Sum of the *Logarithms* of *A* and 10; and 10 × *A* ÷ 10 = *A* whose *Logarithm* is the difference of the *Logarithms* of 10 × *A*, and of 10.

	Numbers.	Logarithms.
Exa.	674800	5.8291751
	67480	4.8291751
	6748	3.8291751
	674.8	2.8291751
	67.48	1.8291751
	6.748	0.8291751

From this it is plain that the *Logarithms* of Numbers in a decuple Progression, differ only in their Indexes, which differ gradually by 1.

A further Application of this to pure Decimal Fractions, you'l find in the following *Probl.* 5.

PROBLEM 2.

To find the *Logarithm* of an Integral Number exceeding the Limits of any Table of *Logarithms*; for *Exa.* exceeding 10,000; to which our common Tables are carried.

RULE. Take as many Figures on the left hand of the given Number as you can find in the Table [*i.e.* 4 of them if the Limit of the Table is 10,000; or 5 if it is 100,000] and in place of the Figures cut off from the right hand, annex o's, so this will express a Number less than the given Number: Again, to the Number express'd by the Figures taken on the left hand add 1, and on the right of the Sum annex as many o's as the Number of Figures cut off the right hand of the given Number, and this will be a Number greater than it: Then take the difference of these two Numbers, which are the one lesser, and the other greater than the given Number; also the difference of the given Number, and the Number lesser than it (which difference consists of the Figure cut off the right hand) and make this Proportion.

As the difference of the Numbers greater and lesser than the given Number
is to the difference of the *Logarithms* [which can be found by the Tables and the preceding second Observation.]
So is the difference of the given Number, and that lesser than it
To the difference of their *Logarithms*, which therefore added to the *Logarithm* of that lesser Number, gives the *Logarithm* of the Number proposed.

Exa.

Exa. To find the *Logarithm* of 123459 from a Table carried only to 10000. The two Numbers lesser and greater than 123459, taken according to the Rule are 123400, and 123500, whose *Logarithms* are 5.0913152, and 5.0916670, for the *Logarithm* of 1234 is 3.0913152; to which add 2, the *Logarithm* of 100 (because 123400 = 1234×100) the Sum 5.0913152 is the *Logarithm* of 123400. Also the *Logarithm* of 1235 is 3.0916670, and so that of 123500 is 5.0916670; and the Proportion is

from	123500	5.0916670	123459
take	123400	5.0913152	123400

As 100, is to .0003518, so is 59 to .00020756 &c. which added to 5.0913152, the Logarithm of 123400, the Sum is 5.09152276 &c. the Logarithm of 123459 nearly. But if we used a Table carried to 100000, the Logarithm sought would be 5.09152278 nearly: And still the more of the Figures of the propos'd Number we have in the Table, the nearer or more exact will the Logarithm be found.

DEMON. The *Reason* of this Rule is founded upon this; That the greater any Numbers are in respect of their Differences, the nearer those Differences are to being proportional with the Differences of their Logarithms, To understand which clearly, *Consider*, 1°. That if we take the natural System 1 . 2 . 3 . 4 . &c. the farther it is continued, the Terms are the nearer to a Ratio of Equality; for $\frac{2}{3}$ is greater than $\frac{1}{2}$, and $\frac{3}{4}$ greater than $\frac{2}{3}$; and so on: Whence, the farther from the beginning we take anytwo adjacent Terms, the nearer they stand together in the infinite Scale of proportionals from 1. Thus since 1 : 2 : : 3 : 6, therefore there fall as many Means betwixt 1 and 2, as betwixt 3 and 6, and consequently more than betwixt 3 and 4; and so it is thro' the whole natural System, considered now as a part of the infinite Progression from 1. But again, 2°, the *Logarithms* of these Numbers are their distances from 1 in the infinite Progression, *i. e.* the number of the intermediate Ratio's; (or they are in proportion to one another as these Numbers;) so that the difference of the *Logarithms* of any two Numbers, is the Number of intermediate Ratios in the infinite Progression betwixt these two Numbers; and from hence, with the preceding Article, it plainly follows, that the differences of the *Logarithms* of the natural System do continually decrease. 3°. The Ratios of the several Terms of the natural System do so grow, as that if we take any three adjacent Terms; the farther they are taken from the beginning, the nearer they approach to being Geometrically proportional: Thus, to 1, 2 a third proportional is 4, which exceeds 3 by 1. To 2, 3 a third ∺*l* is $4\frac{1}{2}$, which exceeds 4 by $\frac{1}{2}$. To 3 : 4 a third ∺*l* is $5\frac{1}{3}$ which exceeds 5 by $\frac{1}{3}$, and so on, the third ∺*l* to any two Numbers adjacent in the Series exceeds the third Term in the Order of the Series, by such an aliquot part of Unity as is denominated by the least of the given Numbers; consequently these excesses become less and less; for this Series does constantly decrease $\frac{1}{2} . \frac{1}{3} . \frac{1}{4}$, &c. *i. e.* every Term of the Series wants less and less of being a true third ∺*l* to the two preceding Terms; and what they want being certain determinate Fractions which grow infinitely little, or less than any assignable Fraction, therefore the Terms grow infinitely near to being ∺*l*. From whence 4°. The differences of the *Logarithms* of the several Terms of the natural System decrease so as to approach infinitely near to being equal. And since the differences of the Terms of the natural System are all equal: Hence it follows, 5°. That taking any three Terms in the natural System (tho' not immediately adjacent) the

the difference of the Extremes is to the difference of the Mean and either Extreme in a Ratio which approaches nearer and nearer to the Ratio of the differences of the Logarithms of the same Terms, the farther these Terms are taken from the beginning, and the nearer they stand together at the same time, *i. e.* the greater the Numbers are with respect to their Differences. And this also shews the Reason why the Rule is more exact and true in finding the *Logarithm* sought, the more of the Figures of the given Number we have in the Table; for, by this means, the Numbers are the greater, and their Differences the lesser.

Observe; If the Number given is compos'd of two or more Numbers within the Table, then the Sum of their Logarithms is that sought: But if it is a *Prime*, we must either use the Method of this *Problem*, or *Problem* 2d: But this Problem, as 'tis easier, so 'tis exact enough for any Numbers we can have Use for, which are not in Tables, these being carried to 10000, and some to 100000.

PROBLEM 4.

To find the Number corresponding to any *Logarithm*, which being the Result of an Operation with Logarithms found in the Table, is not it self found exactly in the Table.

1°. If the Characteristick and first 4 or 5 Figures of the Fractional Part of your Logarithm is found in the Table, that's near enough for common use; and the Number found against such a Logarithm, or that one of several such that is nearest to the given Logarithm, you may take for the Number sought. But if you would have it exacter, or that you cannot find a Logarithm having so many of the Figures of the given Logarithm; Then,

2°. Take the two Logarithms in the Table which are next greater and lesser than the given one, and also their corresponding Numbers, and make this proportion:

As the Difference of the greater and lesser Logarithms
is to the Difference of their corresponding Numbers,
So is the Difference of the given and next lesser Logarithm
to the Difference of their corresponding Numbers. [Which Difference added to the Number corresponding to that lesser Logarithm, makes the Number corresponding to the given Logarithm nearly.]

DEMON. The Reason of this is in the last Problem, of which this is but the Reverse.

Exa. Given this Logarithm, .4669347; the next lesser and greater (in *Sherwin*'s Tables) are .3010300, the Logarithm of 2, and .4771213, the Logarithm of 3; and the Proportion is thus form'd;

from	.4771213	3	.4669347
take	.3010300	2	.3010300

As, .1760193, is to 1, so is .1659047, to .94215 &c. Which added to 2, makes 2.94215 &c. the Number sought nearly,

But we may work another Way, and somewhat more exactly, thus: Seek among the Logarithms, two whose Fractional Parts are next lesser and greater than the given Logarithm (which is a Fraction) these are 4.4669269, the Logarithm of 29304; and 4.4669417 the Logarithm of 29305: But, taking away the Characteristicks, the Fractional Parts are Logarithms to these Numbers, 2.9304 and 2.9305 (according to what has been explain'd in the *Scholium* after *Probl.* 2.) Wherefore the Operation is

from	.4669417	2.9305	.4669347
take	.4669269	2.9304	.4669269

As, .0000148 is to .0001, so is .0000078 to .000052: But the Remainder of the Division is such as makes the nearest answer .000053; which added to 2.9304, makes the Answer or Number sought 2.930453; which is less, and more exact, than the former Answer.

From this *Example*, duly consider'd, there will be no Difficulty to solve any other; wherefore I shall only add this general Direction, *viz*. If the given Logarithm is a *mix'd* Number, we may find a Number answering the Fractional Part of the Logarithm, by this last Method, and then multiply this Number by 1 with as many 0's as there are Units in the Index of the Logarithm; the Product is the Number sought. The *Reason* is obvious from what is explain'd.

3°. If the given Logarithm has a greater Characteristick than any in your Table, seek a Number answering the greatest Index in your Table with the Fractional Part of your Logarithm, by the Method of one of the preceding Articles; Then multiply that Number by 1 with as many 0's as the Number by which the given Characteristick exceeds the greatest in your Table, and that is the Number sought.

But, in some Cases, we can find the Number sought more exactly, thus; Seek a Logarithm whose Fractional Part is nearest to that of the given Logarithm, without regarding the Index; then take the Number corresponding to that Logarithm, and divide it by 1 with as many 0's as the number of Units by which the Index of that Logarithm exceeds the given Index. See an Example of this in the following *Rule* for reducing a Vulgar to a Decimal Fraction.

PROBLEM 5.

To find the LOGARITHM *of a* FRACTION; *and Reversely, a* FRACTION *from its* LOGARITHM.

There are various Ways of finding and expressing the Logarithms of Fractions.

1st Method. Subtract the Logarithm of the Denominator out of that of the Numerator, the Remainder is the Logarithm of the Fraction: But, *Observe*, if it is a proper Fraction, then, because the Numerator is less than the Denominator, so is its Logarithm less than the other's Logarithm, and consequently the Subtraction is impossible: Wherefore we must subtract the Logarithm of the Numerator out of that of the Denominator, and the Remainder, taken negatively, is the Logarithm sought.

Exa. 1. The Logarithm of $\frac{463}{24}\left(=19\frac{7}{24}\right)$ is 1.2853698; for the Logarithm of 463 is 2.6655810; and that of 24 is 1.3802112, which exceeds the other by 1.2853698.

Exa. 2. The Logarithm of $\frac{74}{863}$ is -1.0667791; for the Logarithm of 74 is 1.8692317, and that of 863 is 2.9360108; and their Difference is 1.0667791

DEMON. Let $\frac{A}{B}$ express any Fraction, then is $B : A :: 1 : \frac{A}{B}$, whose Logarithms are therefore in Arithmetical Proportion; *that is*, Log. $A -$ Log. $B =$ Log. $\frac{A}{B} -$ Log. 1; But the Log. of 1 is $= 0$, therefore Log. $A -$ Log. $B =$ Log. $\frac{A}{B}$; And there-

therefore, if A is less than B, the Log. of $\frac{A}{B}$ is their Difference taken negatively: By which is shewn, that the corresponding Number is below Unity as far as the Reciprocal of that Number is above Unity, in the infinite Series of Proportionals, the Logarithm of that Reciprocal being the same Logarithm taken positively. So, the Logarithm of $\frac{74}{863}$ being -1.0667791, the Logarithm of $\frac{863}{74}$ is 1.0667791; For as $\frac{74}{863} : 1 : \frac{863}{74}$ are $\div l$, so $-1.0667791 : 0 : 1.0667791$ are $\div l$.

For the *Reverse* of this Problem, *viz.* If a Negative Logarithm is given to find its corresponding Fraction, find a corresponding Number to the Logarithm, consider'd as positive, and by that Number divide 1, the Quote is the Fraction sought. The *Reason* is plain; for if two Numbers or Logarithms consist of the same Figures, but the one Positive, and the other Negative, their Sum is 0; Also the Product of two reciprocal Fractions is 1; or 1 divided by any Number, makes a Quote, which multiply'd by that Number produces 1; wherefore if 1 is divided by the Number corresponding to any Logarithm taken positively, the Quote is the Number corresponding to the same Logarithm taken negatively.

But the finding the *Fraction* from the *Logarithm* is not so convenient by this Method as by the following.

2d Method. Subtract the Logarithm of the Denominator from that of the Numerator; and if it's a proper Fraction, when you come to the Characteristick or Integral part, subtract that of the Numerator from that of the Denominator (after adding to it the 1 borrow'd in the preceding place, if there was 1 borrow'd) the Fractional part of this Remainder is taken positively, and the Integral part negatively; and the negative Sign set over it, to shew that this part only is negative.

Exa. The Logarithm of $\frac{74}{863}$ is $\bar{2}.9332209$; as below, in the Margin.

Log. of	74	is	1.8692317
	863	is	2.9360108
	$\frac{74}{863}$	is	$\bar{2}.9332209$

DEMON. This is in Effect the same Logarithm as was found by the former Method; for we may take it thus, $.9332209 - 2$, which being resolv'd by Subtraction, is -1.0667791, *viz.* the difference of 2 and $.9332209$ taken negatively. That the Methods must coincide in all Cases, I shall demonstrate *Universally*, thus:

Let any proper Fraction be $\frac{N}{M}$; the Logarithm of N be call'd $A + B$ (A the Characteristick, and B the Fraction); the Logarithm of M be $C + D$ (C the Characteristick, and D the Fraction); Then, by the 1st Method, the Log. of $\frac{N}{M}$ is $A + B - C - D$. or $C + D - A - B$, taken negatively. For the 2d Method, we shall first suppose that D does not exceed B; then the Log. of $\frac{N}{M}$ is by that Method

$$\text{Log. of } N = A + B = A + 1 - 1 + B$$
$$\text{of } M = C + D.$$
$$\text{Log. of } \frac{N}{M} = A + B - C - D. \text{ by 1st Method.}$$
$$\left.\begin{aligned} &= B - D - \overline{C - A} = \\ &= B + 1 - D - \overline{C + 1 - A} \end{aligned}\right\} \text{2d Method.}$$

$B - D - \overline{C - A}$; for we first subtract D from B, then we take A from C, and take this Remainder negatively. which makes $B - D - \overline{C - A} = B - D - C + A = A + B - C - D$, as before. *Again*; If D is greater than B, in this case 1 must be taken from A and

added to B; so that it is $A + B = \overline{A - 1} + \overline{1 + B}$. Now, if we first subtract D from $1 + B$, the Remainder is $B + 1 - D$; and subtracting $A - 1$ (because 1 was before taken from it and added to B) from C, or A from $C + 1$, which is the same thing, the Remainder is $C - A + 1 = C + 1 - A$; which being taken negatively, or subtracted from the former part, (*viz.* $B + 1 - D$) it is $B + 1 - D - \overline{C + 1 - A} = B + 1 - D - C - 1 + A = A + B - C - D$, as before.

SCHOL. As the preceding *Rule* is General, relating to all Fractions, so it comprehends *Decimal* Fractions; and because the Denominator of every Decimal is in this Series 10 . 100 . 1000 . &c. whose Logarithms are pure Integers in this Series 1 . 2 . 3 . &c. Therefore it's evident, that to find the Logarithm of a Decimal Fraction, having found the Logarithm of the Numerator, the Fractional part of it is the Fractional part of the Logarithm sought; and for the Index, apply with a negative Sign the Difference of the Indexes of the Logarithm of the Numerator, and the Logarithm of the Denominator: So the Logarithm of 64 being 1 . 8061800, the Logarithm of .064 must be $\bar{2}$. 8061800; for the Denominator is 1000 whose Logarithm is 3, then $1 - 3 = \bar{2}$: Hence again it is plain, That the Index of the Logarithm of a Decimal Fraction shews in what Place after the Point the first Figure on the left hand of the Numerator stands; and that Distance, therefore, does shew, reciprocally, what the Characteristick of the Logarithm is. And if we take any Decimal, pure or mix'd, *i. e.* a Decimal Fraction, *proper* or *improper*, the General Rule for finding the Logarithm is plainly this, *viz.* Find the Logarithm of the Numerator (*i. e.* of the Number express'd by all the Figures, in order as they stand, neglecting the Point) the Fractional part of that Logarithm is the Fractional part of the Logarithm sought: And for the Index, 'tis the Number which expresses the Distance of the last Figure on the left hand, after the Place of Units of the Integral part, if it's a *mix'd* Decimal; or the Distance of the last Figure on the left hand after the Decimal Point, if it's a *pure* Fraction.

Observe also, That if we take the same Number, and multiply it continually by 10, also divide it continually by 10, whereby we form a Geometrical Progression in the Ratio of 1 to 10, the Logarithms of this Series will have all the same Fractional part, and differ only in their Characteristicks, which will be an Arithmetical Progression differing only by 1: So that when in the descending Series we are come to a Term whose Logarithm has 1 for its Index, the Index of the next must be 0; and all after that are the preceding Indexes, 1, 2, 3, &c. taken negatively, as in the prefix'd Scheme: Which is all but the application of what has been explain'd, particularly of what has been said in the Consequences last deduced from this Method of finding the Logarithm of a Fraction compar'd with what was formerly observ'd in the General *Scholium* 8.

Numb.	Logarithms
25400	4 . 4048337
2540	3 . 4048337
254	2 . 4048337
25.4	1 . 4048337
2.54	0 . 4048337
.254	$\bar{1}$. 4048337
.0254	$\bar{2}$. 4048337
.00254	$\bar{3}$. 4048337

For the *Reverse* of this Method, (*viz.* finding a Fraction from its Logarithm) the *Rule* is, Take the Logarithm as altogether positive, and find the corresponding Number; to which apply a Decimal Point on the left hand of it, so that it's first Figure on the left stand in such a place from the Point as is express'd by the Index of the Logarithm; And this is the Fraction sought, reduced to a Decimal Fraction.

Exam.

Exam. The given Logarithm $\bar{2}.8061800$; the corresponding Number to the Log. 2.8061800 is 640; and, to adjust it to the negative Index $\bar{2}$, it is .0640 = .064:

Observe: If the corresponding Number first found be a mix'd Decimal, the rest of the Work is the same; for we neglect the Point in that first Decimal: So, if the given Logarithm is $\bar{1}.6707096$. the corresponding Number to 2.6707096 is 468.5; and, because of the negative Index, it is .04685

The *Reason* of this Rule is evident from what has been explain'd in the preceding *Scholium.*

COROLL. From this we have learnt the following

Rule *for reducing a* Vulgar *to a* Decimal *Fraction.*

Find the Logarithm of the *Vulgar* Fraction by this *2d Method*, and then find its corresponding *Decimal* Fraction by the *Reverse.*

3d Method. Subtract the Logarithm of the Denominator from that of the Numerator, borrowing (*viz.* 1 from the next place) wherever the Figure of the Subtractor exceeds its Correspondent in the Subtrahend, even to the very last place, tho' the Subtractor be a greater Number than the Subtrahend. The Remainder, or Number thus found, is the Logarithm sought, and is all positive.

Exa. The Logarithm of $\frac{74}{863}$ is 8.9332209, as in the Margin.

Log. of	74	=	1.8692317
of	863	=	2.9360108
			8.9332209

DEMON. The Foundation of this Method, (which gives the Logarithm in a System different from the common one) is this, *viz.* That as a decuple Progression may be taken both ascending and descending from 1; thus, .001, .01, .1, 1, 10, 100, 1000, &c. So we may apply 0 as the Logarithm of any of these Terms we please, whereby the Logarithms of the rest will be, in the ascending Series, 1.2.3.&c. and for the descending Series, they will be −1, −2, −3, &c. *Now* suppose we chuse 1 in the 10th place of Decimals, *viz.* .0000000001, from whence to begin the Logarithms (*i. e.* whose Logarithm we make 0) then is 1 the Logarithm of 1 in the 9th place, or .000000001; and so on till we come to 1 whose Logarithm is 10; Whence the Logarithm of 10 will be 11, the Logarithm of 100 is 12, and so on; So that the Logarithms of all Numbers above .0000000001 are positive; the Indexes of all from .0000000001 to .000000001 being 0; from .000000001 to .00000001 being 1; and so on. But the Logarithms of all Numbers below .0000000001 are negative; or their Indexes at least, according as they are taken by the preceding *1st* or *2d Method.*

Numbers.	Logarithms.
.000000000001	−2
.00000000001	−1
.0000000001	0
.000000001	1
.00000001	2
.0000001	3
.000001	4
.00001	5
.0001	6
.001	7
.01	8
.1	9
1.	10
10.	11
100.	12

Hence again, The Logarithm of 1 being 10, every Index from 10 upwards belongs to the Logarithm of a Number integral or mix'd: And if we take 10 from any Index greater, the remainder shews in what Place after that of Units (of the integral part) the first Figure on the left hand of the corresponding Number stands; But if the Index is less than 10, then, as it belongs to a Decimal Fraction, so, if it's taken from 10, the Remainder shews in

what

what Place after the Point the first Figure on the left of the Numerator stands. And here 'tis to be observ'd, that all *pure* Decimals, whose first Figure on the left of the Numerator stands in any given Place after the Point, belong all to the same Class, or have the same Index in their Logarithm, because whatever number of Figures follow after that Place, they cannot make the whole equal to an Unit in the preceding Place on the left. So all Decimals, whose first Figure of the Numerator, on the left, stands in the first Place after the Point, (as .3, or .47, or .5067) are Intermediates betwixt .1 and 1, and so have 9 for the Index of their Logarithm. All whose first Figure stands in the second Place are intermediate betwixt .01 and .1, and so the Index of their Logarithm is 8; and so on: And this is to be understood, tho' there were an infinite number of Figures, from any Place after the Point, unless that infinite number were all 9, for then the Value of that Infinity is an Unit in the preceding Place on the left, and ought rather to be written thus, .0999 &c = .1; *Thus* then we have a new System of Logarithms, differing from the common ones only in their Indexes, which exceed these, in every Logarithm, by an equal excess of 10; and consequently the Logarithms of Numbers in decuple Progression differ as in the common Logarithms, *viz.* only in their Indexes, which differ gradually by 1.

Now, for the *Reason* of the preceding Rule, consider, that to find the *Logarithm* of a Fraction (which does not fall below .0000000001) as $\frac{74}{863}$, we must argue thus; $863 : 74 :: 1 : \frac{74}{863}$; wherefore the *Logarithm* of the Fraction is the Remainder after the *Logarithm* of 863 is taken from the Sum of the *Logarithms* of 74 and 1: But, by what's now explain'd, the Indexes of these *Logarithms* must, in this new System, be more by 10 than they are in the common form; and, because the *Logarithm* of 1 (*viz.*10) is to be added to that of 74, therefore the Index in the Subtrahend will be 21 (for 11 is now the Index of the *Logarithm* of 74) and that in the Subtractor will be 12, whence that in the Remainder is 8: But because there is 20 added to the Index of the common Logarithm of 74, (which is but 1) and 10 to that of 863 (which is but 2) 'tis the same thing in Effect if we take the Index of the common Logarithm of 863, and add only 10 to that of 74; which is the thing the Rule prescribes, and is therefore just. The like Reason is obvious in all Cases.

Observe again, that 0 may be also apply'd as the *Logarithm* of a Decimal whose Numerator is 1 in the 100th Place after the Point; and then the *Logarithm* of 1 in the 99th Place after the Point will be 1; and so on to 1 Integral, whose *Logarithm* will be 100, and that of 10 will be 101, and so on upwards: Wherefore to the Indexes found in the common form there will be 100 added; and, consequently, if the common Index of the Number wants more than 10 of the Index of the Denominator, we must add 100; Or, it is the same thing if we add 10 gradually from one Place to another, according as the Rule directs.

As to the *Reverse* of this Method, *viz.* From the *Logarithm* of a Fraction to find the Fraction; *Consider*, that as the Index of the *Logarithm* of a *pure* Fraction, which falls not below 1 in the 10th or 100th, &c. Place of Decimals, is some Number below 10 or 100, &c. So, if we take the *Difference* betwixt the Index of the *Logarithm* and the Logarithm of 1, *viz.* 10, or 100, &c. and apply that as the Index to the Fractional part of the *Logarithm*, then find the corresponding Number to that *Logarithm*, and qualifie it again by setting a Decimal Point on the left hand of it, so that the first Figure on the left stand as far from the Point as the Index expresses, we shall have thus the Decimal Fraction sought.

SCHOLIUM. For the further application of this Method, *observe*, that as the Logarithms of Fractions are taken in a System different (at least in their Indexes) from the common ones; so in all Operations with such Logarithms, regard must be had to the Logarithm of Unity, which is now 10, or 100, &c. For *Exa.* If two Fractions are to be Multiplied, and if they are such that their Logarithms are both taken out of the System, wherein the Logarithm of 1 is 10, then take the difference betwixt 10 (the Logarithm of Unity) and the Index of the Sum of these Logarithms; and if that Sum was greater than 10, the difference shews how many places to the left of the 1 whose Logarithm is 0 (*viz.* .0000000001) the first significant Figure on the left of the Decimal sought must stand; and consequently taking this difference from 10, the new difference shews how many places from the point this first Figure stands; but if 10 is greater than the Index of the Sum, the difference is to be taken negatively, and shews that the first Figure on the left of the Decimal stands so many places to the right after the 1 whose Logarithm is 0; so that adding 10 to that remainder, the Sum shews in what place after the point the Decimal begins. *Observe* also in this last case, that you'll find the same Number by subtracting the first remainder from 20. *Exam.* If the Index of the Sum of the Logarithms of two Fractions (taken out this way) is 14, then the corresponding Decimal Fraction begins in the fourth place after the point; but if the Index is 8, the Decimal begins in the 12th place.

Again; tho' it is a certain general Rule, that the Logarithms of two Numbers which are to be Multiplied together, are to be taken out of the same System of Logarithms; yet that is not always necessary, because other Circumstances save it; as an Example or two will explain.

Suppose the Logarithm of one Fraction is 3.5768340 (belonging to that System where in 10 is the Logarithm of 1,) and of another 12.3742067 (belonging to that System where in 100 is the Logarithm of 1;) if we reduce them to one System, the first is 93.5768340, and then their Sum is 105.9510407; from which take 100 (the Logarithm of 1) the remainder is 5.9510407, whose Index 5 taken from 100, the difference 95 shews at what place after the point the correspondent Decimal begins. But suppose we do not reduce the first Logarithms, adding them as they are, the Sum is 15.9510407, from which if we take 10 (the Logarithm of 1 in the one System) the remainder is 5.9510407 as before: The *Reason* is obvious; for as here one of the Indexes is 90 less than in the other Method, so there is 90 less subtracted from the Sum, which must give the same difference.

If an Integer and Fraction are to be Multiplied, then if we consider the Integer Fraction ways, making 1 the Denominator, taking its Logarithms out of any System, it will be the same as when we take it out of the common System as a whole Number; because what we add to the common Index is taken away again by substracting the Logarithm of the Denominator 1; and therefore from the Sum of the Logarithms of an Integer, taken out of the common System, and the Logarithm of a Fraction taken by this third Method, we are to take the Logarithm of 1 (as it is in the System out of which the Logarithm of that Fraction was taken) the remainder is the Logarithm of the Product; the difference of whose Index and 10, if it's greater than 10, or their Sum if its less, shews in what place from the Point the Decimal begins.

Exa. Suppose the Logarithm of any Integer is 4.3720679 (taken out of the common System, wherein the Logarithm of 1 is 0) and the Logarithm of some Fraction is 3.2578006, (taken out of the System where 10 is the Logarithm of 1) the Sum of these Logarithms is 11.6298685 from which take 10 (the Logarithm of 1) the remainder is 1.6298685 whose Index taken from 10 the remainder is 9, shewing that the corresponding Decimal begins in the ninth place after the Point.

But at last *observe* as to both this and the preceeding Method, that if Fractions are given Terms in a Question which is to be solved by Multiplication or Division, we need not,

not, in order to the Operation by Logarithms, take out the Logarithms of these Fraction; separately; but, considering the Method of Working with the given Numbers, we shall need only to take out the Logarithms of the several Terms as whole Numbers, and apply them to one another by Addition, so as to have at last but one Simple substraction; and then if a greater Logarithm is to be taken out of a lesser (which gives the Logarithm of a Fraction) we may do it either by the preceding second or third Method, and then find the corresponding Fraction in the manner directed: The following Questions will illustrate this: For

I shall finish this Chapter with a few Examples in the Practice of Logarithms for a further illustration of the Rules; where you'l also find some further useful Instructions

In Multiplication.

Exa. 1. To Multiply 39674 by 4685, I find the Sum of their Logarithms; which having a greater Index than any in the Tables, I find the nearest Logarithm to 8.2692156 which is 4.2692093, whose corresponding Number is 18587; and because the Index of the Logarithm whose corresponding Number I want is 8, I multiply the Number found by 10000 and it is 185870000, which is less than the true Product, this being 185872620. If we we had Tables carried to a greater extent than 101000 (which is the extent of *Sherwin's* Tables) then we should find the Product true to more Places.

Logarithms.	
of 4685 =	3 . 6707096
of 39674 =	4 . 5985060
of 185870000 =	8 . 2692156

Exa. 2. To find the Product of 268 by $\frac{57}{342}$ I add the Logarithm of 268, to that of 57 and from the Sum take the Logarithm of 342, and the Number corresponding to the remainder nearest is 44.667; the true Product being 44.666, &c. having 6 circulating *in Infinitum*; so that the Product found is a very little more than true. I have also wrought it another way, to shew the correspondence of both; thus, I take the Logarithm of $\frac{57}{342}$ by the second Method of the preceding Problem, which is $\bar{1}.2218488$ which added to the Logarithm of 268, the Sum is the same Logarithm as was found by the other Method.

Logarithm.	
of 268 =	2 . 4281348
57 =	1 . 7558749
Sum	4 . 1840097
of 342 =	2 . 5340261
of 44 . 667 =	1 . 6499836
of $\frac{57}{342}$ =	$\bar{1}$. 2218488
of 268 =	2 . 4281348
Sum	1 . 6499836

Exa. 3. To Multiply $\frac{23}{478}$ by $\frac{7}{59}$ I add the Logarithms of 7 and 23, also those of 59 and 478; and substracting this Sum from that, as directed in the second Method of the preceding Problem; the remainder is the Logarithm sought, *viz.* $\bar{3}.7565459$ whose correspondiing Number I find, thus, I seek a Logarithm whose Fraction is nearest to .7565459, and this I find to be .7565448, and the corresponding Number is 57088, which qualified according to the Index $\bar{3}$, is .0057088; which exceeds by a little the true Product, for this is .0057008, &c.

Logarithms.	
of 7 =	0 . 8450980
23 =	1 . 3617278
7 × 23 =	2 . 2068258
59 =	1 . 7708520
478 =	2 . 6794279
59 × 478 =	4 . 4502799
$\frac{7 \times 23}{59 \times 478}$ =	$\bar{3}$. 7565459, diff.

whose Value reduced is this Decimal .0057088.

In

In Division.

Exa. 1. To Divide 4762 by 24, I substract the Logarithm of this from the Logarithm of that, the remainder is 2.2975782 to which the nearest Logarithm in the Table (setting aside the Index) is .2975636 which is the Fractional part of the Logarithm of 19841; but applying the Index 2, the corresponding Number is 198 . 41.

Logarithms.		
of 4762	=	3 . 6777894
24	=	1 . 3802112
198.41	=	2 . 2975782, diff.

Exa. 2. To Divide 74568 by 4.37; having found the Logarithm of 437, whose Fractional part is .6404814, the Index due to make it the Logarithm of 4.37 is 0; therefore taking the Logarithm of 4.37 from that of 74568 the remainder is a Logarithm, whose nearest in the Table has for its corresponding Number 17103, which exceeds the true Quote, for this is 17063.615, &c.

Logarithm.		
of 74568	=	4 . 8725525
4.37	=	0 . 6404814
17103	=	4 . 2320711

Exa. 3. To divide 5670 by $\frac{37}{456}$; because the Quote is equal to $5670 \times 456 \div 37$, therefore I add the Logarithm of 5670 to that of 456, and from the Sum take the Logarithm of 37; the Logarithm in the Table which is nearest to the remainder, has for its correspondent Number 69879, the true Quore being 69878.918, &c. wanting very little of the former.

Logarithms.		
of 5670	=	3 . 7535831
456	=	2 . 6589648
Sum		6 . 4125479
of 37	=	1 . 5682017
of 69879	=	4 . 8443462

To Divide a Fraction by a Fraction is multiplying the Dividend by the Reciprocal of the Divisor, and the Operation by Logarithms is the same therefore as in Multiplication.

In finding Proportionals.

Exa. 1. To find a 3d in Geometrical Proportion to these 14 : 359, I take the Logarithm of 359, and from the double of this Logarithm take the Logarithm of 14, the remainder is a Logarithm whose nearest in the Table has for its correspondent Number 9205.8, which is the 3d Proportional sought nearly, this being 9205.78, &c.

Logarithms.		
of 359	=	2 . 5550944
its double		5 . 1101888
of 14	=	1 . 1461280
of 9205.8	=	3 . 9640608 diff.

Exa. 2. To find a fourth Proportional to these 24 : 367 . 29 : : 5348 . 6 from the Sum of the Logarithms of the second and third Terms, I take the Logarithm of the first 24; the nearest Logarithm to the remainder has for its correspondent Number 81854, which exceeds the true fourth a little, this being 81853.63, &c.

Logarithms.		
of 5348 . 6	=	3 . 7282401
of 367 . 29	=	2 . 5650091
Sum		6 . 2932492
of 24	=	1 . 3802112
of 81854	=	4 . 9130380

Exa. 3. To find a fourth Proportional to these $\frac{13}{14} : \frac{35}{37} :: \frac{5}{9}$: By the common Rules this is had by this Operation, *viz.* $5 \times 25 \times 14 \div 9 \times 37 \times 13$, wherefore having added together the Logarithms of 5, 25, 14, also the Logarithms of 9, 37, 13, I take this Sum from the other: And seeking a Logarithm in the Table whose Fractional part is nearest to 6066504, its corresponding Number is 40425, and because the Index of my Logarithm is $\bar{1}$, therefore the Number sought is .40425, which is the true fourth Proportional, true in all these Figures.

Logarithms.	
of 5 =	0 . 6989700
of 25 =	1 . 3979400
of 14 =	1 . 1461280
Sum	3 . 2430380
of 9 =	0 . 9542425
of 37 =	1 . 5682017
of 13 =	1 . 1139434
Sum	3 . 6363876
of .40425 =	$\bar{1}$. 6066504 diff.

Observe. If in *Multiplication, Division,* or finding a third or fourth Proportional, any of the Terms is a mixt Number, either reduce it to an improper Fraction, or the Fractional part to a Decimal, and then proceed.

For Involution.

This being no other than Multiplication; the practice of it by Logarithms is the same also as that for Multiplication.

For Extraction of Roots.

Exa. 1 To extract the Square Root of 1156: I find its Logarithm to be 3.0629578, whose half is 1.5314789, and the correspondent Number is 34, which is exactly the Square Root of 1156.

Exa. 2. To find the fifth Root of 32768; I find its Logarithm 4.5154499, and the fifth part of this is .9030899, *&c.* the nearest Logarithm to which in the Table is .9030900 whose correspondent is 8, the true fifth Root of 32768.

Exa. 3. To find the Cube Root of 13839, I take its Logarithm which is 4.1411047, whose third part is 1.3803682, *&c.* The Logarithm whose Fractional part is nearest to this is .3803741, and its corresponding Number is 24009, but because of the Index 1, it is 24.009, which is an excessive Root, for the Cube of this is 13839.578, *&c.* The Integral part of which Root 24, is the Root of the greatest Integral Cube, which is contained in 13839.

Observe. If the given Number whose Root is sought is greater than any Number in your Table, use this Method; take a Number lesser, which is a Power of the proposed Order, by which divide the given Number; if the Integral Quote is a Number within your Table (and if it is not, you must chuse another Divisor of the same kind that will bring the Quote within the Table) seek the proposed Root of the Quote, and multiply it by the Root of the Divisor, the Product will be the Root sought, or near to it.

Exa. 4. To find the Cube Root of 262144 because it's greater than can be found in the Table, I divide it by 8, (the Cube of 2) which gives for a Quote precisely 32768, whose Logarithm is 4.5154499, and the third part of this is 1.5051499, and the Number corresponding to that Logarithm which is the nearest to this in the Table, is 32; which multiplied by 2 (the Cube Root of the Divisor 8) produces 64, the true Cube Root sought.

The *Reason* of this Rule you have in *B* 3, *Theor.* 2d. For if any Number is a Power, as A^n and if it is divided by a Similar Power B^n, the Quote is a Similar Power, whose Root is the Quote of the Roots of the Dividend and Divisor. So $A^n \div B^n = \overline{A \div B}^n$ are $A \div B \times B = A$, that is, the n Root of the Quote of $A^n \div B^n$ multiplied by B, the

the Root of the Divisor B^n, produces A, the n Root of the Dividend A^n. If the Dividend is not a Rational Power, or the Divisor is not an aliquot part of it; you can only expect to find a Root nearly true: But as Involution is easier than Evolution, having found such a Root as your Logarithms will give, prove it by actual Involution, and by an Allowance for what it errs, and one or two Trials, you may bring it near enough for common Applications: And in the Extraction of high Roots, where the common Rules prove very tedious, *This* will with much less Trouble bring out a Root sufficiently near.

APPENDIX, *shewing the* Reason *of the* RULES *given for finding the number of Terms in a Geometrical Progression.*

See *Probl.* 4, 6, 9, *Chap.* III, *Book* IV.

In *Problem* 4th having the Extremes a, l, and Ratio r, to find the number of Terms n. The Rule given by Logarithms is this; $n - 1 = \frac{\text{Log. } l - \text{Log. } a}{\text{Log. } r}$; the *Demonstration* of which is this, $l = ar^{n-1}$ (*Cor.* 6, *Probl.* 3, *Chap.* III.) whence $r^{n-1} = \frac{l}{a}$; And consequently, Log. r^{n-1} = Log. $\frac{l}{a}$. But, by *Prob.* 5. preceding, Log. $\frac{l}{a}$ = Log. l — Log. a; and by the 3d Fundamental Rule of *Logarithms*, Log. $r^{n-1} = \overline{n-1} \times$ Log. r; Wherefore $\overline{n-1} \times$ Log. r = Log. l — Log. a; and, lastly, $n - 1 = \frac{\text{Log. } l - \text{Log. } a}{\text{Log. } r}$

In *Problem* 6th the Rule is $n - 1 = \frac{\text{Log. } l - \text{Log. } a}{\text{Log. } \overline{s-a} - \text{Log. } \overline{s-l}}$; the Reason of which is this; By the preceding part of that Problem it is shewn, that $r = \frac{s-a}{s-l}$; Whence, Log. r = Log. $\frac{s-a}{s-l}$ = Log. $\overline{s-a}$ — Log. $\overline{s-l}$, which being put in the preceding Rule for Log. r, makes the present Rule.

In *Problem* 9 we have this Rule; $n - 1 = \frac{\text{Log. } l - \text{Log. } \overline{rl+s-rs}}{\text{Log. } r}$. The Reason of which is this; 'Tis there shewn, that $a = rl + s - rs$; so that this Rule is only putting Log. $\overline{rl+s-rs}$ for Log. A, in the Rule of *Problem* 4.

We have also this Rule, $n - 1 = \frac{\text{Log. } \overline{rs+a-s} - \text{Log. } r - \text{Log. } a}{\text{Log. } r}$; the Reason of which is, that $l = \frac{rs+a-s}{r}$, whence Log. l = Log. $\frac{rs+a-s}{r}$ = Log. $\overline{rs+a-s}$ — Log. r; which is put in place of Log. l, in the Rule of *Problem* 4.

CHAP. VI.

Of the Combinations *of* *Numbers.*

DEFINITIONS.

I. *Combinations* of Things are, the Various Ways a number of Things may be taken and join'd together, either in respect of the Order of the Whole, or the Choice of a number of Particulars out of the Whole. But this will be more clearly understood by the Species into which *Combinations* are distinguish'd, viz. *Permutations*, *Elections*, and *Compositions*.

II. *Permutations*, or *Changes*, (or, as some call them, *Alternations*) are such Combinations of any number of Things wherein respect is had to the Order of the Whole, either as to Place or Succession, thus. (1°) In regard of Place: Any number of Things being propos'd, the number of different Ways these Things may be dispos'd in an equal number of determin'd Places, so that they shall never be all in the same Places, is call'd their *Changes* (in respect of Place). *Exa.* Suppose 6 Things *A*, *B*, *C*, *D*, *E*, *F*, are to be dispos'd in 6 Places: This may be done various Ways, according to the different Places every one may possess, Regard being still had to the Whole; *i.e.* if any two, or more of them change Places, that makes a new *Alternation* or Order of the Whole, tho' all the rest remain unchang'd.

(2°) In regard of *Succession*: The different Ways several Things may be taken or order'd in Succession one after another, are also call'd *Changes*, or *Alternations*, as to Order of Succession, depending upon the taking of *A*, or *B*, or any one of them, 1st or 2d, &c. And as the taking any one of them 1st or 2d, &c. may be call'd putting them in the 1st, 2d, &c. Place of the Succession, this shews the Coincidence of these two Ways of ordering Things, as to the Number of *Changes*; For they are both reducible to one Notion of Place, either as it relates to *Space*, which is more strictly call'd *Place*; or to *Time* and *Succession*; which, as to the Number of Changes, is the same; for *Places* cannot be better distinguish'd than by numbering them 1st, 2d, 3d, &c. and the Order of Succession of Things is distinguishable no other Way, than by marking which Thing is 1st, 2d, 3d, &c.

III. *Elections* or *Choices* are Combinations which regard not the Order of the Whole, but the Way of taking a particular Number out of the Whole. *Thus*, Suppose a lesser number of Things is to be taken out of a greater, and we are at liberty to take them out of any Part of the Whole; the number of Ways this may be done, so that some (one at least) shall be different in every Choice or Combination, is call'd the *Choices* of that number of Things in the other. *Exa.* If 4 Men are to be drawn out of 100, the number of Ways this can be done, so as some one of them shall be a different Man, is the *Choices* of 4 Men (or any other Things) in 100.

COROLLARIES.

1st. The *Choices* of 1 in any Number is equal to that Number; and any Number can be taken out of it self but once, or one Way.

2*d*. If

2*d*. If any Number N is equal to two Numbers, $A + B$, the *Choices* of A and B in N are equal: For, since the one being taken, the other is left; then as many Choices as you can take away of the one, so many you leave of the other.

3*d*. If two Numbers differ by 1, as A and $A + 1$, the Choices of A in $A + 1$ is equal to $A + 1$; because the Choices of 1 in $A + 1$ is $A + 1$, and the Choices of A and 1 are equal.

IV. *Compositions* are limited Elections. Thus, conceive two or more different Setts (or Systems) of Things, containing each the same, or a different number of Things; then suppose we are to chuse out of the whole a number of Things, either equal or unequal to the number of Setts, so that we take some Part out of every Sett [if possible, *i. e.* if the number of Setts be not greater than the number to be chosen; for then we may take any Choice of a number of the Setts equal to the number to be elected] the number of Choices thus limited is call'd the *Compositions* of that number of Things out of that number of Setts. *Exa.* Suppose 16 Companies of Men, 16 Men may be drawn out of these various Ways, taking only 1 Man out of each Company, and the number of Choices we can make with this Limitation of 1 out of each Company, is the *Compositions* of 16 in the 16 Companies.

THEOREM 1.

If any Number N is resolv'd into two Parts, $A + B$, the *Changes* of N or $A + B$ are equal to the Product of the *Changes* of A, and of B; and the *Elections* of A (or B, which are equal by *Corol.* 2. to *Defin.* 2d) in $A + B$.

Exa. If the Changes of 4 be 24, and of 6 be 720, and the Elections of 4 in 10 $(= 4 + 6)$ be 210, then the Changes of 10 will be $24 \times 720 \times 210 = 2158800$.

DEMON.

$$\underbrace{\overbrace{\cdot\ \ \cdot\ \ \cdot\ \ \cdot}^{A\,=\,4}\ \ \ \overbrace{\cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot}^{B\,=\,6}}_{A\,+\,B\,=\,10}$$

Conceive a number of Places equal to $A + B$ represented by Points set in a Row, as in the Margin; wherein there are distinguish'd a Number equal to A on the left, and the remainder equal to B on the right: It's certain that in every one of the *Changes* of $A + B$ Things in all these Places, some one particular Election of a number of Things equal to A must possess that individual number of Places equal to A, which lies first on the left hand. Suppose any one Election of A Things to possess those Places, it's plain they can continue there so long, and no longer, than till all their Changes in these Places be join'd with all the Changes of the remaining B Things in the remaining Places on the right hand; and these will make so many different Changes of the whole, *viz.* a Number equal to the Product of the Changes of A into the Changes of B: But then every Election of A Things out of the whole will possess these A Places, that lie first on the left hand, as often, or in as many different Changes of the whole, as the first Election did; and when every Election of A has possess'd these A Places as often as possible, (*i. e.* in as many different Changes of the whole as the Product of the Changes of A and B) then all the Changes of the whole are finish'd. Consequently they are the continual Product of the Changes of A, and of B, and the Elections of A (or B) in $A + B$: Which may be express'd in Characters, thus; $ch : A + B = ch : A, \times, ch : B, \times$ Elect. A in $\overline{A + B}$.

THEOREM 2d.

Let any Number N be equal to two others $A + B$, the number of those different Alternations of the whole N, in which the Part A will possess the same number of certain determin'd.

min'd Places, is equal to the Product of the number of Alternations of A by those of B.

Exa. If the Changes of 6 are 720, of 2 if they are 2, and of 4 they are 24; then a certain Choice of 4 Things will possess a certain Choice of 4 Places, 48 ($= 24 \times 2$) times, or in 48 different Changes of the whole.

D E M O N. It's plain that the Number A may possess an equal number of determin'd Places as long as while all their Alternations in these Places be join'd with all the Alternations of the remaining Things B in the remaining Places, and no longer.

Observe; If you ask how long these A Things will keep these determin'd Places without changing in them, then the Number is that of the Alternations of B.

T H E O R E M 3d.

If a Number n ($= a + b$) is to be elected out of a greater N, the Number of different Elections, in which a certain Choice of a Things will cast up, is equal to the number of Elections of b in $N - a$.

Exa. If the Elections of 6 in 10 are 210; and of 4 in 8 ($= 10 - 2$) they are 70; then chuse any 2 out of the whole 10, and that Choice will come up with 70 different Choices of 6 in 10.

D E M O N. Take away any Choice of a from N, the Choices of b in the Remainder $N - a$ being join'd with that Choice of a, make all the Choices of $a + b$ in which this particular Choice of a is concern'd.

T H E O R E M 4th.

The *Elections* of any Number A in another greater than it, N, are equal to the Sum of the Elections of A and of $A - 1$ in $N - 1$.

D E M O N.

$$\overbrace{\cdot\;\;\underbrace{\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot\;\;\cdot}_{N-1}}^{N}\;\; \&c.$$

Conceive all the Units of the Number N to be dispos'd in a Row, and one of them to be taken off, from the left hand, so that there remain $N - 1$ on the right: It's evident that the Choices of A in $N - 1$ are a part of the Choices of A in the whole N; and 'tis as plain that, having these, we want none of the Choices sought (*viz.* of A in N) but those in which the Unit taken off is concern'd (or makes one of the Units chosen). And it's again plain, that these are had by joining that Unit with all the Choices of $A - 1$ in $N - 1$, because that Unit being join'd to $A - 1$, makes the Number A; and being join'd to all the Choices of $A - 1$ in $N - 1$, makes all the Choices of A (in N) in which that Unit is concern'd: Which Number therefore being added to the Choices of A in $N - 1$, makes the whole Choices of A in N.

P R O B L E M 1st.

To find how many *Alternations* or *Changes* any Number of different Things is capable of,

R U L E. Take the Natural Series of Numbers from 1 (*viz.* 1, 2, 3, *&c.*) up to the given Number; multiply them together, the last Product is the Answer.

Exa. 1. The Changes of 3 Things are $6 = 1 \times 2 \times 3$, represented as in the Margin, by 3 Letters, A, B, C.

Exa. 2d.

A, B, C	*B, C, A*
A, C, B	*C, A, B*
B, A, C	*C, B, A*

Exa. 2d. The Changes of 8 Things are, $40320 = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$, So that if there are 8 Men in a Company, they may change Places so, that the Order of the whole shall be varied 40,320 different Ways.

DEMON. If the Rule is true in any one Case, (*i. e.* of a Number N, it's true of the next above, or of $N+1$ Things; and consequently of all above: But 'tis true of 2 Things, A, B, whose Changes are only $2 = 1 \times 2$, for they can be order'd only thus, $A\,B$ or $B\,A$; therefore 'tis true of all Numbers. What remains then to be demonstrated is this; That if the Rule is true of N Things, it's true also of $N+1$ Things, which is demonstrated thus:

By *Theorem* 1, the Changes of $N+1$ are $=$ *ch*: $N \times$, *ch*: 1, $\times$ elect: N in $\overline{N+1}$. But the Changes of 1 are only 1; *that is*, one Thing can be taken but one Way: And the Elections of N in $N+1$ are $N+1$ (*Corol.* 3, *Defin.* 2) therefore the Changes of $N+1$ are $=$ *ch*: $N \times \overline{N+1}$. But the Changes of N, according to the Rule, are $N \times \overline{N+1} \times \overline{N-2} \times$ &c. $\times 1$, (or $1 \times 2 \times 3 \times$ &c. N); and if this is right, then the Changes of $N+1$ are $\overline{N+1} \times N \times \overline{N-1}$ &c. $\times 1$. (or $1 \times 2 \times 3$, &c. $\times \overline{N \times 1}$.) which being also according to the Rule, this is therefore right.

Or, this Article may be demonstrated independently of *Theor.* 1, from the nature of Alternations only, *thus*; In every *Change* of $N+1$ Things, some one Thing must possess the 1st Place, and there it may continue till the remaining N Things change Places as oft as possible; each of which Changes join'd with that One in the 1st Place, makes so many different Changes of the Whole: And, since any One of the Whole may possess the 1st Place as oft, it follows that the Changes of N Things multiply'd by the Whole number of Things, $N+1$, gives all the different Changes of $N+1$. But *ch*: $N = N \times \overline{N-1}$ &c. $\times 1$. Therefore *ch*: $N+1 = \overline{N+1} \times N \times \overline{N-1}$, &c. $\times 1$; which is the Rule.

SCHOL. I. We have learnt how to find the Number of Changes of any Number of Things; but if it should also be requir'd actually to take them all out, or represent them, for Example, by Letters; there is one Certain Method of proceeding, by which we can go thro' the whole with the greatest Ease and Distinctness, so as to run no hazard (or the least possible) of omitting any Change, or taking any one oftner than once. This Method will be made clear by a few Examples.

Exa. 1. For 2 Things *A, B*, the Changes are these 2, *A B, B A*.

Exa. 2*d*. For 3 Things, *A, B, C*, the Changes are 6, which you see already taken out; only, to save superfluous writing, they may be order'd as in the Margin; where, because every Letter possesses the 1st Place twice, *viz.* till the remaining two have chang'd twice, therefore I write that Letter down but once in the 1st Place, supposing it to belong to the 1st Place of the next Change, which is left not fill'd up.

A B C	*B A C*	*C A B*
C B	*C A*	*B A*

Exa. 3*d*. For 4 Things, *a, b, c, d*, the Changes are 24, as they are here represented; where every Letter possesses the 1st Place 6 times, *viz.* till the remaining 3 have chang'd 6 times, whose *Changes* are order'd the same way as in the preceding Example.

a b c d	*b a c d*	*c a b d*	*d a b c*
d c	*d c*	*d b*	*c b*
c b d	*c a d*	*b a d*	*b a c*
d b	*d a*	*d a*	*c a*
d b c	*d a c*	*d a b*	*c a b*
c b	*c a*	*b a*	*b a*

Exa. 4th. For 5 Things, *a*, *b*, *c*, *d*, *e*, their Changes are 120 ($= 24 \times 5$) taken as here repreſented. Where *Obſerve*, that becauſe the Changes of 4 are 24, ſo, in taking out theſe of 5, every Letter muſt poſſeſs the 1ſt Place 24 times, *i. e.* till the remaining 4 Letters make 24 Changes; which are taken out according to the Method of *Exa. 4th.*

a b c d e	*b a c d e*	
e d	*e d*	
d c e	*d c e*	
e c	*e c*	
e c d	*e c d*	
d c	*d c*	
c b d e	*c a d e*	
e d	*e d*	
d b e	*d a e*	
e b	*e a*	
e b d	*e a d*	&c.
d b	*d a*	
d b c e	*d a c e*	
e c	*e c*	
c b e	*c a e*	
e c	*e a*	
e b c	*e a c*	&c.
c b	*c a*	
e b c d	*a c d*	
d c	*d c*	
c b d	*c a d*	
d b	*d a*	
d b c	*d a c*	
c b	*c a*	

I have carried the Work no farther here than to 48 Changes, *viz.* while the two firſt Letters, *a*, *b*, poſſeſs the 1ſt Place each 24 times; the reſt are eaſily conceiv'd by theſe.

By theſe *Examples* the Method for any other Number may be eaſily underſtood, one depending always upon the preceding: So, if there were 6 Things whoſe Changes are $720 = 120 \times 6$ each of them muſt poſſeſs the 1ſt Place 120 times, *viz.* till the remaining 5 make their 120 Changes. And, *Obſerve*, that as every Letter has the 1ſt Place as oft as the Changes of the remaining Number, ſo, while it poſſeſſes the 1ſt Place, the Letter next it in the 1ſt Change of thoſe wherein it has the 1ſt Place poſſeſſes that next (or 2d) Place as oft as the number of Changes of the remaining Letters after this one; and then the next Letter is advanc'd into that 2d Place; and ſo on, till they are all ſucceſſively in the 2d Place. The ſame is to be obſerv'd of the 3d and 4th, &c. Places: Then, when all the Letters after the 1ſt have poſſeſs'd the 2d Place, a new Letter is advanc'd into the 1ſt Place, and ſo the Changes proceed with that Letter in the 1ſt Place as they did before.

Or, if we trace the Order from the right hand to the left, then *obſerve* that the two Letters on the right hand (in the 1ſt Order of the given Letters) having chang'd twice, a new Letter is advanc'd into the 3d Place (counting now from right to left) and what was laſt in the 3d Place is put in the 2d Place; and this new Letter in the 3d Place is there till the 2 on its right hand change twice, and then the 1ſt Letter (on the right) is advanc'd to the 3d Place: Then, when the firſt 3 Letters have thus poſſeſs'd the 3d Place, each of them twice, *i. e.* as oft as the Changes of the remaining 2, a new Letter is advanc'd into the 4th Place (by making the Letters in the 3d and 4th Places change) and there it continues till the remaining 3 Letters make all their Changes: And ſo on till all the Letters are advanc'd to all the Places from the right to the left Hand. An attentive Conſideration of the preceding Examples will make all this very clear.

But *Obſerve* again, that the Number of Changes grow ſo faſt upon the Series of Numbers, that the *Changes* of a ſmall number of Things can never be all repreſented. For *Example*: The Changes of 10 are 3628806; and allowing a Man to take out 300 of them every Hour, it would coſt him 304 Days to finiſh them all, tho' he works at it Night and Day without Interruption: But if we only double the Number of Things, *i. e.* take 20, the Changes are 187,146,308,321,280,000; ſo great a Number, that if a Man could take out 500 of them every Hour (which yet I doubt any Man could do) it would take him upwards of 42 thouſand million of Years to finiſh them all. For, divide the Changes by 500, the Quote is 374292616642560 Hours; which divided by 24, quotes 15595525693440 Days, which makes 42,727,330,666 Years 70 Days.

SCHOL. 2. In this *Problem* the Things to be *chang'd* are ſuppos'd to be ſo many diſtinct Individuals; which, tho' equal or alike in ſome reſpects, yet are diſtinct and different from one another in that reſpect upon which the Variety of *Change* depends; and ſo are capable

capable of a real Difference, and Variety of Order : But if two, or more, of them are the ſame, or like in that reſpect upon which the *Change* depends, ſo that they admit of no Variety among themſelves; Then the Number found by the former Rule muſt be corrected. I ſhall firſt explain this Likeneſs and Difference of Things.

Whatever Likeneſs we ſuppoſe among Things, while we conſider them as only numerically different, this is a ſufficient Foundation for all the Variety of Order or Change in the preceding Problem : But if we make the Subject of the Change any Thing which they have all in common, or which is common to any Number of them, the Caſe is different. For *Example* ; Suppoſe 3 Letters, whereof 2 of them are the ſame as to Sound, as *A*, *A*, *B*, theſe are not capable of all the Variety, in reſpect of the Order and Succeſſion of Sounds that 3 different Letters have, becauſe the two *A*'s having no Variety of Sound, admit of no Change betwixt themſelves, as two different Letters do : So the Changes here are only 3, viz. *A A B*, *A B A*, *B A A* ; whereas 3 different Letters have 6 Changes. But if we take two different Characters, as *A*, *a*, and make the Change regard only the Places of different Characters, without regard to Sound, then it's the ſame thing what Sounds they repreſent, they are 3 different Things as to Shape, and ſo have all the Variety of any 3 different Things as to Order of Place.

Take another *Example:* Suppoſe 4 Bells, whereof 2 of them have the ſame Note or Tone in Muſick; then, if we conſider the Changes theſe 4 Bells are capable of in the ſucceſſion of their Sounds, as Notes of Muſick, they have not ſo many as were they all different Notes ; becauſe the two that have the ſame Note cannot change with one another, and ſo it's no matter which of them is firſt ſtruck, it makes the ſame ſucceſſion of Notes. Indeed if we conſider 'em only as 4 Sounds, emitted from 4 diſtinct Bodies, they are in this reſpect capable of all the Variety of any 4 different Things, tho' they had all one Note ; but the Variety in this reſpect is not to be perceiv'd by the Ear, unleſs the ſpecifick difference of the Sounds be all different, and then the Changes may be ſaid to turn upon that, otherwiſe the Changes of them can only be mark'd by different Names to the 4 Bells.

PROBLEM 2d.

To find all the *Changes* of any number of Things, whereof 2 or more are the ſame, in that reſpect upon which the Change depends :

RULE. Find the Changes of the Given Number, and alſo the Changes of that Number of them which are the ſame, or like, by *Probl.* 1 ; divide the former by this, the Quote is the true number of different Changes. But if there are more than One Part of the Given Number that conſiſt of Things like among themſelves, (One Part being ſtill different Things from another) then take all the Parts (*i.e.* all their Numbers) which conſiſt of like Things among themſelves; find the Changes of each of theſe Numbers by *Probl.* 1. then multiply them continually together, and by the Product divide the Changes of the Given Number found by *Probl.* 1, the Quote is the true Number ſought.

Exa. 1. Of 6 Things, whereof 3 are the ſame, the Changes are 120 : For the Changes of 6 different Things are 720, thoſe of 3 are 6 ; then $720 \div 6 = 120$.

Exa. 2. Suppoſe 8 Notes of Muſick whereof 3 are the ſame, and 2 are the ſame, but different from the former 3, and both different from the remaining 3, as, *fa*, *fa*, *fa*, *ſol*, *ſol*, *la*, *mi*, *fa*, the Variety in the ſucceſſion of theſe 8 Notes, is 3360; thus, the Changes of 8 different Notes are 40320; of 2 there are 2, and of 3 there are 6; then $2 \times 6 = 12$, and $40320 \div 12 = 3360$.

Demon. Suppose any number of Changes $N = A + B$, by *Theor.* 1, $ch{:}\,N = ch{:}\,A \times ch{:}\,B \times$ Elect. A (or B) in N; therefore if any Number, as A, of these Things has but one Order, (as when they are like Things, or the same in that respect upon which the Variety depends) in this Case $ch{:}\,N = ch{:}\,B \times$ Elect. A (or B) in N: that is, the Number found by *Probl.* 1. is to be divided by $ch{:}\,A$, consider'd as different Things. Again: if there is another part of the given Number all like Things, as, suppose $B = C + D$, and the Number C are like Things, then, by what's already shewn, the $ch{:}\,B$, taken by *Probl.* 1. must be divided by $ch{:}\,C$; Consequently, the $ch{:}\,N$ (by *Probl.* 1.) are to be first divided by $ch{:}\,A$, to correct the Error arising from A being like Things: and this Quote again divided by $ch{:}\,C$, to correct the Error arising from C being like Things, and so on, however many Parts are like Things: But it's the same to divide $ch{:}\,N$ continually by any Numbers one after another, or all at once, by the continual Product of these Numbers. Wherefore the Rule is true.

P R O B L E M 3d.

To find the *Elections* of any lesser number of Things out of a greater number of Things all different.

Rule. Take the Series 1, 2, 3, &c. up to the Number to be elected, and multiply them continually together; then take a Series of as many Terms, decreasing by 1, from the Number out of which the Election is to be made, and multiply them continually together: Divide this Product by the former, the Quote is the Number sought.

Exa. 1. The Choices of 2 in 6 are $15 = \dfrac{6 \times 5}{1 \times 2} = \dfrac{30}{2}$

Exa. 2. The Choices of 4 in 9 are $126 = \dfrac{9 \times 8 \times 7 \times 6}{1 \times 2 \times 3 \times 4} = \dfrac{3024}{24}$

Universally, the Choices of A in B are express'd thus;

$$\frac{B \times \overline{B-1} \times \overline{B-2} \times \overline{B-3}\,.\,\&c. \times \overline{B-A-1}}{1 \times 2 \times 3 \times 4 \times .\,\&c. \times A}$$

Demon. Suppose $B = A + D$, then, by *Theor.* 1, $ch{:}\,B = ch{:}\,A \times ch{:}\,D \times$ Elect. A (in B); Hence Elections A (in B) $= ch{:}\,B \div \overline{ch{:}\,A \times ch{:}\,D}$. Now, by *Probl.* 1, $ch{:}\,B = B \times \overline{B-1} \times \overline{B-2}$ &c. $\times 1$; And 'tis evident, that in this Series there must be one Term equal to D (since D is less than B) And therefore this Series may be thus express'd, $ch{:}\,B = B \times \overline{B-1}$, &c. $\times D \times \overline{D-1}$ &c. $\times 1$. But since $B = A + D$ and $D = B - A$, therefore the Term of this Series next above D or $B - A$ is $\overline{B - A + 1} = \overline{B - A - 1}$; therefore the Series may be also thus express'd, $ch{:}\,B = B \times \overline{B-1}$ &c. $\times \overline{B - A - 1} \times D \times \overline{D-1}$ &c. $\times 1$. Now the $ch{:}\,D = D \times \overline{D-1}$ &c. $\times 1$, therefore $ch{:}\,B$ being divided by $ch{:}\,D$, the Quote is $= B \times \overline{B-1}$ &c. $\times \overline{B - A - 1}$; also $ch.\,A = 1 \times 2 \times 3$ &c. $\times A$. And, the last Quote being divided by this, the Quote is $\dfrac{B \times \overline{B-1}\ \&c. \times \overline{B-A-1}}{1 \times 2\ \&c. \times A}$, which is $= ch{:}\,B \div \overline{ch{:}\,A \times ch{:}\,D} =$ Elect. A in B. Which being according to the Rule, 'tis therefore right.

But we may also demonstrate this Rule independently of *Theor.* 1, from the Nature of *Elections* only, thus:

If the *Rule* is true in any one Case, (*i. e.* of the Elections of A in B) it's therefore true of the next Case, or of $A + 1$ in B. But the Rule is true of the Choices of 1 in any Num-

Number B, which, according to the Rules, are $\frac{B}{1} = B$, which is the true Number, (as has been obſerv'd in *Corol.* 1, *Definit.* 3.) Therefore the Rule is true of the Choices of 2 in B, and conſequently of 3, and every other Number: What is to be demonſtrated then is this; That becauſe the Rule is true of any Number A in B, it's therefore true of $A+1$ in B; which I thus demonſtrate.

Take any one Choice of A out of B, there remain $B-A$ things; and if each Unit of this Remainder be ſeverally combin'd with that Choice of A, we ſhall hereby have as many Choices of $A+1$ as the Number $B-A$ expreſſes. Again; If we take every other Choice of A, and combine them with every Unit of their ſeveral Remainders, for every one we ſhall have as many Choices of $A+1$ as $B-A$ expreſſes; *i. e.* in the whole a number of Combinations of $A+1$, equal to the Product of the Choices of A (in B) by $B-A$; which, according to the Rule, is $\frac{B \times \overline{B-1}\ \&c. \times \overline{B-A+1}}{1 \times 2\ \&c. \times A} \times B-A = \frac{B \times \overline{B-1}\ \&c. \times \overline{B-A+1} \times \overline{B-A}}{1 \times 2\ \&c. \times A}$. But now theſe Combinations of $A+1$ that we have thus ſuppos'd are not all different; and to find how many of them are ſo, take any one Election of $A+1$, and call it the firſt; then conceive all the Elections of A that are in this firſt Election of $A+1$ (which are ſo many of the Elections of A in B) to be combin'd with each Unit of their Remainders in B, theſe make ſo many of the preceding Combinations of $A+1$; and it's plain, that with each of theſe Elections of A (in this firſt Election of $A+1$) ſo join'd with each Unit of their Remainders in B, there will ariſe one Combination of $A+1$, Coincident with this firſt Election of $A+1$; for each Election of A, in this Election of $A+1$, being join'd with the remaining Unit in the ſame Election of $A+1$, coincides with it: Wherefore as many Elections of A as are in this firſt Election of $A+1$ (which are in Number $A+1$) ſo many of the preceding Combinations of $A+1$ are coincident, and therefore not different Elections of $A+1$.

But the ſame is, for the like reaſon, true of every other really-different Election of $A+1$ in B; ſo that for every really-different Election of $A+1$ in B, there are $A+1$ Combinations of $A+1$ Things taken in the preceding Work, which are coincident Elections of $A+1$ Things; wherefore we muſt divide that firſt Number of Combinations of $A+1$ Things by $A+1$ (which is done by multiplying the Diviſor of that Operation by $A+1$) the Quote is the true Number of the different Elections of $A+1$ in B, which is this, $\frac{B \times \overline{B-1}\ \&c. \times \overline{B-A+1} \times \overline{B-A}}{1 \times 2\ \&c. \times A \times A+1}$, exactly according to the Rule.

In the following *Scholium* you ſee yet another Way of demonſtrating this Rule.

SCHOL. I. The Elections of any Number in any greater may be found and diſpos'd in a Table (where they may be afterwards had by Inſpection) which may be carried on *in infinitum*; whereof you have here a Specimen; the Conſtruction of which is obvious, every Column being made of the Sums of the preceding ſo far, or of the Sum of the preceding Terms of the ſame and the preceding Column. The Numbers to be elected ſtand on the head of the Table, the firſt Column being the Numbers out of which the Elections are to be made, and the ſeveral Numbers of theſe Columns ſhew the Elections of the Number on the head in the correſponding Number of the firſt Column.

Exa. The Elections of 7 in 10 are 120, found in the Column under 7, and againſt 10 in the firſt Column.

TABLE *for the Elections of* Numbers.

Numbers to be elected.

Numbers out of which the Elections are made.	2	3	4	5	6	7	8	9
.1								
1	Numbers of Elections.							
2	1							
3	3	1						
4	6	4	1					
5	10	10	5	1				
6	15	20	15	6	1			
7	21	35	35	21	7	1		
8	28	56	70	56	28	8	1	
9	36	84	126	126	84	36	9	1
10	45	120	210	252	210	120	45	10
11	55	165	330	462	462	330	165	55
12	66	220	495	792	924	792	495	220

The Construction and Use of this Table being thus explain'd, I shall next *demonstrate*, that it contains the true number of Elections, according to the *Rule* for using it, thus:

If 1, a, b, c, &c. represent the numbers of Choices of any Number n, in the several Terms of the progression of Numbers from n upwards, then the sum of this Series of *Choices*, viz. 1, $1+a$, $1+a+b$, &c. are the several Choices of $n+1$ in the several Terms of the progression of Numbers from $n+1$ upwards. This is plain from *Theor.* 2, which proves, that the Choices of a in n are the sum of the Choices of a and $a-1$ in $n-1$; *i. e.* that the Choices of $n+1$ in any number of the Series from $n+1$ is the sum of the Choices of $n+1$ and n in the preceding lesser Number; But the Choices of n are in the Series 1, a, b, c, &c. and the Choices of $n+1$ in $n+1$ are 1; Therefore the Choices of $n+1$, in all the Numbers greater, are in the Series 1, $1+a$, $1+a+b$, &c.

$$\frac{n : n+1 : n+2 : n+3}{1 : a : b : c} \quad \&c.$$
$$1 \quad : 1+a : 1+a+b$$

But again; The Choices of 1, in any Number, are equal to that Number; *i. e.* are the natural Series 1, 2, 3, 4, &c. consequently the Choices of 2 in the Series of Numbers from 2, are the Sums of the preceding Series; and the Choices of 3 in the Series of Numbers from 3, are the Sums of the last Series of Choices: Which makes exactly the preceding Table of *Elections*.

Observe now, That this Table of Elections is the same as the Table of *Triangular* Numbers explain'd in *Chap.* 2, § 2. (only differently dispos'd) where it is shewn, that the a Triangular of the b Order, (or the b Triangular of the a Order) taken from a Series of Units, is $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$ &c. $\times \frac{n-\overline{a-2}-b}{a-1}$ (See the *Schol.* after *Probl.* 1, *Ch.* 2.) Now, in this Table of *Elections*, if the Numbers to be elected are compar'd with the Numbers out of which the Election is to be made, the difference of them is always 1 less than the Place of the Number of Elections in its proper Column: Thus, the Difference of 4 and 10 is 6, and the Elections of 4 in 10 are 210, the 7th Term of the Series of Elections of 4: But the Elections of any Number in another are the same as those of its Difference from that other. Also the several Columns of this Table

Table are the ſeveral Orders of Triangular Numbers, which becauſe as Triangular Numbers they are reckon'd from a Series of Units, the Number of the Order is 1 more than the Number on the head of the Table: And, becauſe alſo the Place of the Number of Elections, in its proper Column, is 1 more than the Difference of the Number on the head of the Table, and the Number of the firſt Column out of which it is to be elected; Therefore 'tis plain, that the Elections of any Number $a - 1$ out of another n ($=$ Elections of $n - \overline{a-1}$ out of n) are the ſame Numbers as the $n - \overline{a-2}$ (or b) Triangular of the a Order; which, by the Rule of Triangulars, is $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times$ $\times$ &c. $\times \frac{n - \overline{a-2}}{a-1}$, and which, according to the Rule of Elections, expreſſes the Elections of $a - 1$ in n; ſo that the Rule being good for Triangulars, 'tis good alſo for Elections.

Or, if we compare this Table of Elections with the Table of Coefficients of the Powers of a Binomial Root, (See *Book* 3, *Ch.* 2, § 1.) 'tis the very ſame, wanting the Column of Units: But the $n - \overline{a-2}$, Coefficient of the n Power, is $1 \times \frac{n}{1} \times \frac{n-2}{1}$ &c. $\times$ $\times \frac{n - \overline{a-2}}{a-1}$, the ſame as the Elections of $n - \overline{a-1}$ in n ($=$ Elections of $\overline{a-1}$ in n) which therefore are $\frac{n}{1} \times \frac{n-1}{2} \times$ &c. $\times \frac{n - \overline{a-2}}{a-1}$, according to the Rule of Elections.

Now at laſt *Obſerve*, That as the Demonſtration of the Rules of Triangular Numbers or Coefficients is a Demonſtration of the Rule of Elections, ſo the ſame Rule being demonſtrated for Elections, from the Conſideration of Elections only, is alſo a Demonſtration of the Rule for Triangulars or Coefficients; becauſe they are the ſame Numbers under all theſe different Views.

SCHOL. 2. If 'tis requir'd to take out all the Elections of any Number out of a greater, the Certain and Regular Method of doing it will be eaſily underſtood by the following Examples.

a b c	*a d e*
a b d	*b c d*
a b e	*b c e*
a c d	*b d e*
a c e	*c d e*

Exa. 1*ſt.* The Elections of 3 out of 5 Things, *a*, *b*, *c*, *d*, *e*, are 10, as in the Margin, which are taken out thus: I take out the firſt 3 Letters as they ſtand in order, *a b c*; then I put another in the firſt place (on the right) ſucceſſively in the order of the Letters, till there is not another behind; then I put a new Letter in the ſecond place (on the right) keeping it there till I change all the Letters in order that are in the firſt place; and then I put a new Letter in the third place, keeping it there till I change all the Letters in the ſecond place as oft as poſſible, *i. e.* ſo as there remain enow behind to make out the Number; and with each of theſe in the ſecond place I change all thoſe in the firſt place; and ſo on, if there are more Things elected; as the following Example will further clear.

Exa. 2*d.* The Elections of 4 in 7 Things: *a*, *b*, *c*, *d*, *e*, *f*, *g*; are 35, *viz.*

In all 35.

The Order of theſe, carefully conſider'd, is ſufficient Direction for any other Caſe. I ſhall only add this Obſervation in the laſt *Example*; That when *g* comes in the firſt place

place (on the right) the Letter in the second place is chang'd, and there it stands till *g* comes again in the first place, and then it is chang'd again. and so on till *f* comes in the second place; and then the Letter in the third place is chang'd, and so it stands till the same Changes as before fall upon the first and second places, *i. e.* till *f*, *g* come together in the first and second places; and then the Letter in the third place is again chang'd; and so it goes on till *e* comes in the third place; and then the fourth place is chang'd; and so on till such a Letter comes in the last place, that what are behind in the Order of the Letters do just make up the Number elected.

SCHOL. 3. The Things out of which an *Election* is to be made are suppos'd to be all different, else the Number found by this *Problem* must be corrected. For example; If we suppose that out of 4 Notes of Musick, 2 are to be elected; and, that 2 of the 4 are equal, or in the same degree, as *fa fa*, *sol*, *la*, then it is plain that we cannot make as many Choices of 2 Notes, that shall be all different Choices, as we could do out of 4 different Notes; for in 4 different Notes we have 6 Choices of 2; but here we have but 4, viz. *fa*, *fa*; *fa*, *sol*; *fa*, *la*; *sol*, *la*. Now, this being a Limitation upon the Circumstances of the Election, is a kind of *Composition* (which is a limited Election) and the Rule for it will be better understood after the General Rules for *Composition*: and, till these are explain'd, I refer it.

PROBLEM 4th.

To find the Sum of all the Choices of every Number that is in any Given Number of Things all different, (*i. e.* the Sum of the Choices of 1, and of 2, and of 3, &c. in any Number *N*) without finding the Choices of any of these particular Numbers.

RULE. Find the Sum of a Geometrical Progression proceeding from 1 in the Ratio 1 to 2, as 1 . 2 . 4 . &c. whose Number of Terms is *N*, the Given Number out of which the Elections are to be made; That Sum is the Number sought. Or, Find such a Power of 2 whose Index is *N*; subtract 1 from that Power, the remainder is the Sum or Number sought.

Exa. The Sum of the Elections of every Number that are in 12, is, $1 + 2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024 + 2048 = 4095 = 2^{12} - 1$

DEMON. Suppose only 2 Things *a*, *b*, all the Choices here are only 3, *viz.* *a* or *b*, or *a b*; if we join another Thing, to make the whole 3, *a*, *b*, *c*, then 'tis plain, that the preceding Choices in *a*, *b*, are so many of the Choices in *a*, *b*, *c*, and we want no more of them but these wherein *c* is concern'd, which are only it self, and its Combinations with all the preceding (set in the 3d Column). Again; join another, making of the whole 4 Things, *a*, *b*, *c*, *d*; the Combinations already found in *a*, *b*, *c* are so many of these sought, and we want only these in which *d* is concern'd, which are only it self, and its Combinations, with all the preceding (set in the 4th Column). The same way does the Work go on, by joining one Thing more, for ever.

a .	*b* .	*c* .	*d*
	ab .	*ac*	*ad*
		bc	*bd*
		abc	*cd*
			abd
			acd
			bcd
			abcd

But it is plain, that the Number of Combinations or Choices in the several Columns are in Geometrical progression from 1, in the Ratio 1 to 2; for the first Column has but one, and every following Column has as many Terms as the Sum of all the preceding, and one more; because the Thing on the head of the Column is join'd with all the preceding Terms: And this is the Property of a Geometrical Progression beginning with 1 in the Ratio of 1 : 2; for the Sum of such a Progression is $\frac{rl - a}{r - l}$; *r* being the Ratio, *l* the

greatest,

greateſt, and a the leſſer extreme; but if r is 2, and $a=1$, then is $\frac{rl-a}{r-1}=2l-1$ ſo that every Term is 1 more than the Sum of all the preceding. Wherefore the Sum of all theſe Columns, *i. e.* of a Geometrical progreſſion, as in the Rule, is the Number ſought.

SCHOLIUM 1. If ſeveral of the things given are the ſame, or like (as explained upon the preceding *Problem*) this Rule is to be corrected; and how to do it you'll find afterwards.

2d. As we have ſhewn a Correſpondence betwixt the Coefficients of the Powers of a Binomial Root, and the Number of Elections of one Number out of another; ſo if we purſue the Compariſon, we learn that the Sum of all the Coefficients after the firſt of any one Power, is the Sum of all the Elections of every Number given, in a Number equal to the Index of that Power; for *Example*, The Coefficients of the fourth Power are 1, 4, 6, 4, 1; and the Elections of 1, 2, 3, 4 ſeverally in 4, are 4, 6, 4, 1; and ſo in any other Caſe, as is clear by compariſon of the Table of Coefficients with that of Elections, which is the very ſame, only wanting the Column of Units, which are Coefficients of the firſt Term in every Power. Wherefore it follows that the Sum of all the Coefficients, after the firſt of any Power is the Sum of as many Terms of a Geometrical progreſſion proceeding from 1, in the Ratio 1 : 2. or taking in the firſt it is the Power of 2, whoſe Index is that given; as it has been alſo formerly ſhewn in Book 3. Again, taking the Rule for the Sum of the Coefficients, as it is already demonſtrated, then the Rule for the Elections is demonſtrated from the Correſpondence; yet it was fit to demonſtrate it alſo from the nature of Elections.

PROBLEM 5.

To find the *Compoſitions* of any Number in an equal Number of ſets, the things being all different.

Rule. Multiply the Number of things in every ſet, continually into one another; the Product is the anſwer.

Exa. 1. Suppoſe four Companies, in each of which there are nine Men; to find how many ways nine Men may be choſen, one out of each Company; the anſwer is $6561=9\times 9\times 9\times 9$.

Obſerve. In all Caſes where the Number in each ſet is equal, the anſwer is always ſuch a Power of that Number whoſe Index is the Number of Sets, or of other things to be choſen; which is here the ſame.

Exa. 2. Suppoſe four Companies; in one of which there are ſix Men, in another eight, and in each of the other two, nine; in this caſe the Choices (by compoſition) of four Men are $3888=6\times 8\times 9\times 9$.

DEMON. Suppoſe only two ſets, it's plain that every Unit of the one ſet being combined with every Unit of the other, make all the Compoſitions of two things in theſe two ſets; and the Number of them is plainly the Product of the Number in the one ſet, by that in the other. Again, if there are three ſets, the Compoſition of two in any two of them being combined with each Unit of the third one, makes all the Compoſitions of three: That is, the Compoſitions of two, in any two of the ſets being multiplied by the Number of the remaining ſet, produces the Compoſitions of three in the three ſets; which is plainly the continual Product of all the three Numbers in the three ſets; and becauſe it's no matter in what order ſeveral Numbers are continually multiplied, therefore it's no matter which two ſets we had ſuppoſed to be firſt taken. For the ſame Reaſon the Rule is good for 4, 5, &c. ſets. Wherefore univerſally, if a, b, c, &c. repreſent the Num-

Numbers in so many sets of things; the Compositions of as many things out of these Sets are, $a \times b \times c \times$, &c.

Problem 6th.

To find the Compositions of any Number in a greater Number of sets, *i. e.* how to chuse a Number of Things out of a greater Number of sets, taking but 1 Thing out of 1 set (the things being all different.)

Rule. Find the Elections of the Number to be chosen (n) in the Number of sets (N) by *Probl.* 3; then find the Compositions of n, in each of these Elections; by the last *Problem*; the Sum of all these Compositions is the answer. But observe, that as the Number of Individuals in each set are supposed to be equal or not, there is a difference in applying the last part of the Rule, thus:

(1°.) If the Number of Individuals is equal in each set, call it m; then the Compositions in every Election of the sets are equal; and by *Probl.* 5. they are m^n; and this being multiplied by the Elections of n in N, gives the Number sought.

Exa. To find the choices of 3 Men out of 5 Companies, which have each 4 Men, so as to take out 1 Man out of 1 Company, the Elections of 3 in 5 are 10, and the 3d Power of 4 is 64, then $64 \times 10 = 640$ the Number sought. But as there are 20 Men in the whole 5 Companies, if the Choice were unlimited, the Number would be 1140.

(2°.) If the Numbers in each set are not equal, we must actually mark out all the Elections; and take the compositions of each separately, because they cannot be all equal; their Sum is the Number sought.

Exa. Suppose 5 Companies whose Numbers are $a = 2$, $b = 3$, $c = 4$, $d = 5$, $e = 6$, out of which are to be chosen by Composition 3 Men: The Elections of 3 in 5 are 10, which being marked out, with the Compositions of 3 in each, the total Sum is 490, as here.

		Compositions of 3 in each set.
a, b, c	$= 2, 3, 4$	24
a, b, d	$= 2, 3, 5$	30
a, b, e	$= 2, 3, 6$	36
a, c, d	$= 2, 4, 5$	40
a, c, e	$= 2, 4, 6$	48
a, d, e	$= 2, 5, 6$	60
b, c, d	$= 3, 4, 5$	60
b, c, e	$= 3, 4, 6$	72
c, d, e	$= 4, 5, 6,$	120
		490 total.

Demon. The Reason of every part of this Rule is obvious from the nature of the thing, and the preceding Rules; and need not be further insisted upon.

Problem 7th.

To find the Compositions of any Number N in a lesser Number of Sets n, of things all different, so as some part be taken out of each set.

Rule. Let the sets be represented by the Numbers in each; then (1°.) distribute the Number N into as many parts as there are Units in n, and do this as many ways as possible; after which (2°.) make out all the Alternations of the Terms or Parts in every distribution, and (3°.) compare each Alternation (taking their Terms in one order, as from left to right) with the Numbers of the several sets, (taken in the same order). These Alternations in which any Term is greater than the Number of the corresponding set are to be rejected; for this shews that these parts cannot be taken out of the sets in that order; and for every other Alternation, you are to do thus. (4°.) Find the

the Elections (by *Probl.* 3.) of every Term of the Alternation out of the Number of the correspondent Set; which multiply continually together, the Product is so many of the Choices sought: Then lastly, the Sum of all these Products, made for every Alternation of every Distribution, is the Number sought.

Exa. 1. Suppose two Companies, the one of 5, and the other of 7 Men, out of which are to be chosen 4 Men, so as a part be taken out of each: How many Choices are there? The Answer is 455; found thus,

(1°.) The sets are 5 and 7, and the Number 4 has two distributions into parts, *viz.* 1 + 3 and 2 + 2. (2°.) The Alternations of these 2 distributions are only these, 1 + 3, 3 + 1, 2 + 2, then (3°.) Comparing these Alternations with the sets, there is none of them to be rejected; and (4°.) the Elections of their parts in the Correspondent sets being taken and Multiplied, and the Products added together, as you see done below, the total Sum is 455.

Sets. 5 : 7	Sets. 5 : 7	Sets. 5 : 7	
Altern. 1 + 3	Altern. 3 + 1	Altern. 2 + 2	Prod. 175, 70, 210
Elect. 5 × 35 = 175.	Elect. 10 × 7 = 70.	Elect. 10 × 21 = 210	455 sum

Exa. 2. There are three Companies of 5, 6, 7 Men; out of which 5 Men are to be chosen by Composition; the Answer is 6055, found as in the following Work.

1°. The distributions of 5 into 3 Parts are only $1 + 1 + 3 = 5$, $1 + 2 + 2 = 5$

2°. The Alternations of the distribution 1 + 1 + 3 are these 1, 1, 3 : 1, 3, 1, and 3, 1, 1; of the distribution 1 + 2 + 2, they are 1, 2, 2 : 2, 1, 2, 2, 2, 1, each of which can be taken in order out of the sets, 5, 6, 7; and the Elections found, multiplied and added, make in all 6055.

Sets. 5 : 6 : 7	Sets. 5 : 6 : 7	Sets. 5 : 6 : 7
Altern. 1 + 1 + 3 = 5	Altern. 1 + 3 + 1	Altern. 3 + 1 + 1
Elect. 5 × 6 × 35 = 1050.	Elect. 5 × 20 × 7 = 700	Elect. 10 × 6 × 7 = 420.

Sets. 5 : 6 : 7	Sets. 5 : 6 : 7	Sets. 5 : 6 : 7	Products
Altern. 1 + 2 + 2	Alt. 2 + 1 + 2	Alt. 2 + 2 + 1	1050, 700, 420, 1575, 1260, 1050
Elect. 5 × 15 × 21 = 1575.	Ele. 10 × 6 × 21 = 1260.	Ele. 10 × 15 × 7 = 1050.	Sum 6055

Exa. 3. Suppose 9 Men are to be chosen by Composition out of 3 Companies, of 4, 5, 6 Men.

Distributions of 9 into 3 parts.				Alternations of these 7 distributions.	Sets. 4 : 5 : 6		4 : 5 : 6		4 : 5 : 6
1 +	1 +	7		1st.	1 , 1 , 7	3d.	1 . 3 . 5	5th	2 . 2 . 5
1	2	6			1 , 7 , 1		1 . 5 . 3		2 . 5 . 2
1	3	5			7 , 1 , 1		3 . 1 . 5		5 . 2 . 2
1	4	4		2d.	1 , 2 , 6		3 . 5 . 1	6th	2 . 3 . 4
2	2	5			1 , 6 , 2		5 . 1 . 3		2 . 4 . 3
2	3	4			2 , 1 , 6		5 . 3 . 1		3 . 2 . 4
3	3	3			2 , 6 , 1	4th	1 . 4 . 4		3 . 4 . 2
					6 , 1 , 2		4 . 1 . 4		4 . 2 . 3
					6 , 2 , 1		4 . 4 . 1		4 . 3 . 2
								7th	3 . 3 . 3

If each of these Series of alternations (of the several distributions of the Number 9) is compared in order with the Numbers of the Sets, 4, 5, 6 (in order to which I have set these over the Alternations) then we find that none of the 1st distribution are useful, because 7 is greater than it's Correspondent Sett; of the 2d distribution, the 2d, 4th, 5th, 6th alternations are also useless, because 6 corresponds to the Sett 4 and 5, out of which it cannot be taken; also the 5th, 6th alternation of the third distribution; and the 3d of the 5th, are useless, all the other alternations serve: And *Observe*, That the distribution which has one part greater than the Number of any of the Setts, is useless, and therefore we need not take out it's alternations; nor the alternations of any other distribution, wherein any part is greater than the Correspondent Sett out of which it should be elected; and therefore taking only the useful distributions and alternations, the whole Work is as follows; where I have set the Elections not under, but in a line with the the Numbers elected; thus, the Elections of 1, 2, 6 (in 4 . 5 . 6) are 4, 10, 1.

Distributions.	Setts. 4 . 5 . 6	Elections.		Setts. 4 . 5 . 6	Elections.		Setts. 4 . 5 . 6	Elections.
1st	1 . 2 . 6	4×10×1 = 40	3d	1 . 4 . 4	4× 5×15=300	7th	2 . 3 . 4	6×10×15=900
	2 . 1 . 6	6× 5×1 = 30		4 . 1 . 4	1× 5×15= 75		2 . 4 . 3	6× 5×20=600
2d	1 . 3 . 5	4×10×6 = 240		4 . 4 . 1	1× 5× 6= 30		3 . 2 . 4	4×10×15=600
	1 . 5 . 3	4× 1×20 = 80	4th	2 . 2 . 5	6×10× 6=360		3 . 4 . 2	4× 5×15=300
	3 . 1 . 5	4× 5×6 = 120		2 . 5 . 2	6× 1×15= 90		4 . 2 . 3	1×10×20=200
	3 . 5 . 1	4× 1×6 = 24	5th	3 . 3 . 3	4×10×20=800		4 . 3 . 2	1×10×15=150
		534			1655			2750

The Sum of all these Products is $534 + 1655 + 2750 = 4939$; the total Elections of 9 in the 3 Companies, whereas a free Choice out of 15, the Sum of the 3 Companies is 5005.

Observe, That tho' this Work be indeed tedious in most Cases, yet it's vastly easier than the actual marking out of all the Choices out of the Setts; which still would require the distributing the Number to be chosen, and then electing.

DEMON.

DEMON. Some Part of the Number to be elected must be taken out of each Sett, which is the reason of distributing that Number into as many Parts as there are Setts, and doing this as many Ways as possible: But then the Parts of these Distributions may be apply'd variously to the Setts, which is the reason of taking all their Alternations and applying them to the Setts; for 'tis manifest that each of these different Applications or Ways of taking the Parts of the Number to be elected, among the Setts, must make a different Choice. But *again*, In each of these we are also to consider in what Part of the corresponding Sett it is taken, for that also must make a difference; so that the Elections of every Number in its corresponding Sett being found, 'tis plain that we can join any Choice of the Number taken out of one Sett with the Choice of the Numbers taken out of the other Setts; and consequently the Variety, in this respect, is the continual Product of the Numbers of Choices of the several Parts of any Distribution out of its corresponding Sett. Therefore, *lastly*, the Sum of all these Products is the Number sought

PROBLEM 8th.

To find the *Elections* of any Number n in another Number N, of Things which are not all different, in that respect which makes the Variety of Choice.

RULE 1°. Of the Number N take the several Parts that are like Things, and reckon each of these Numbers as so many Setts; and all the remaining Things in N as another Sett: Mark each Sett by its Number, and also by some Letter, (because different Setts may have the same Number, but, being Setts of different Things, they must have another Distinction) and because all the Setts, except one, are suppos'd to consist each of like Things, (tho' different Setts consist of different Things) distinguish that one by some Mark. Then,

2°. Find all the Elections of n that can be had in that Sett which consists of Things different, and for each of the other Setts reckon but one Election: But, if the number of Things in any of these Setts is less than n, there's no Election. Then,

3°. Find and mark out all the Elections of every two Setts, of every 3, 4, &c. (as far as the Number n) by their Letters and Numbers; And,

4°. Find the Compositions of n in each of these Choices of different Setts, by *Probl.* 5, 6, and 7.

Lastly, The Sum of all these Compositions, and of the Elections in *Article* 2d, is the Number sought.

Exa. 1. Suppose 4 to be elected out of 11, whereof 5 are like Things, and 6 different: The Choices of 4 in 5 are 5; but the 5 being here all like Things, there is but one Choice; then in the 6 different Things there are 15, both which make 16. Next there is but one Combination to be made of the Setts, because there are but two of them. Now, the Distributions of 4 into 2 Parts, with their Alternations, are $1+3$, $3+1$, and $2+2$; and the Choices of these out of two Setts make in all 41, (as in the annex'd Operation) where the Sett 5 being of like Things, the Choices in it are reckon'd but 1; And to these 41 add the preceding 16, the Sum is 57, the Number sought.

a	*b*
a	*c*
a	*d*
a	*e*
a	*f*
	g

Setts.	5 . 6		
	1 . 3	Elect.	$1 \times 20 = 20$
	3 . 1		$1 \times 6 = 6$
	2 . 2		$1 \times 15 = 15$
			41

But if each of the Setts contains like Things, then the Answer would be only 5; For there is 1 Choice in each Sett, and 1 for every Alternation of the Parts of 4; and if the Things to be elected are 6, then the Elections are 6; for there is no Election in the one

Sett,

Sett, and there is but 1 in the other: Then the Distributions of 6 are $1 + 5$, $2 + 4$, $3 + 3$, and the two former have 2 Alternations, but each of them has but 1 Election in the two Setts, which make in all 5; and this, with the former 1 in the Sett 6, is in all 6.

Exa. 2*d.* To find the Choices of 4 out of 18, whereof 6 are like Things, and 5 also like, and 7 different: The Answer is 201, which is found as in the following Work; for the understanding whereof nothing needs be said, but that the Sett 7, which consists of Things all different, is distinguish'd by a Cross.

$A . B . \overset{+}{C}$	$A . B$	$A . \overset{+}{C}$	$B . \overset{+}{C}$	$A . B . \overset{+}{C}$
$6 . 5 . 7$	$6 : 5$	$6 . 7$	$5 . 7$	$6 . 5 . 7$
$4 . 4 . 4$	$1 + 3$ Elect. $1 \times 1 = 1$	$1 + 3$ Elect. 1×35	$1 + 3$ Elect. 1×35	$1 + 1 + 2$ Elect. $1 \times 1 \times 21$
$1 + 1 + 35 = 37$, the	$3 + 1$ Elect. $1 \times 1 = 1$	$3 + 1$ Elect. 1×7	$3 + 1$ Elect. 1×7	$1 + 2 + 1$ Elect. $1 \times 1 \times 7$
sum of the Choices of	$2 + 2$ Elect. $1 \times 1 = 1$	$2 + 2$ Elect. 1×21	$2 + 2$ Elect. 1×21	$2 + 1 + 1$ Elect. $1 \times 1 \times 7$
4 in the several Setts.	3	63	63	35

Then $37 + 3 + 63 + 63 + 35 = 201$, the Number sought.

Demon. The *Reason* of this Rule consists all in this; That when any Number n is to be elected out of any Number N without limitation; and if the Number N, is any how distributed into Parts $A + B + C$, &c. then it's certain that every Choice of n out of the whole Sum must either be taken out of some 1 of these Parts, or out of some 2, or 3, &c. of them (proceeding to a Number of Parts equal to n, for we cannot distribute it into more Parts than n has Units, and find some Part of n in each Part of N.) Then, for the Manner of taking n out of these Setts or Parts of N, the Reason is contain'd in the *Problems* refer'd to.

Problem 9th.

To find the Sum of all the *Elections* of every Number in any Number N, which are not all different Things, without finding the Elections of any particular Number,

Rule. Separate the Number N into Setts, as in the last *Problem*; then find the Sum of a Geometrical Series from 1 in the Ratio of $1 : 2$, whose Number of Terms is the Number of Things in that Sett which are all different Things: Next take one of the Setts which consists of Things all like, and by its Number multiply the preceding Sum, and to the Product add that preceding Sum, and also the Multiplier: This last Sum multiply by the Number of another Sett of like Things, and to the Product add the last Sum, and also the Multiplier: and thus go on, multiplying the last Sum by the Number of another Sett of like Things, adding the last Sum and Multiplier to the Product till you have gone thro' all the Setts: The Sum of all these Sums is the Answer of the Question.

Exa. To elect every Number out of 9 Things whereof 2 are like, and 3 are like, and 4 different, the Answer is 253. For, the whole Elections in the 4 different Things are (by *Probl.* 4.) $1 + 2 + 4 + 8 = 15$; Then $15 \times 2 = 30$. and $30 + 15 + 2 = 47$. then $47 \times 3 = 141$. and $141 + 47 + 3 = 191$. Lastly, $15 + 47 + 191 = 253$, the Number sought.

Demon.

Demonstration.

$$\overbrace{a:b}:\overbrace{c:d}:\overbrace{e:ee}:\overbrace{f:ff:fff}$$

ab . ac . ad . ae . aee . fa . aff . afff
bc . bd . be . bee . fb . bff . bfff
abc . cd
abd
acd &c.
bcd
abcd

However many different Things there are in the 1st Sett, the Elections fought in that Number are according to the Rule; which is demonstrated in *Problem* 4th. Then, if all the following Things were also different, the same Operation would go on; but if 2 or more of them are the same, the combining these with each of the preceding, will produce only the same Elections: But then taking 1, or 2, or 3, &c. of these like Things (to the greatest Number of them) what you take consider'd by it self, and also join'd with each of the preceding Elections, makes so many new Elections: And thus going thro' them all, we have the whole Election fought, according to the Rule, as the annex'd Scheme will easily shew.

Problem 10th.

To find the *Compositions* of a Number out of an equal number of Setts; When the Things in the several Setts are different (*i. e.* there is nothing in one Sett which is in any other) but the Individuals of the same Sett such, that 2 or more of them are the same or like Things, which may obtain in any one or more Setts.

Rule. Each Sett, whose Individuals are all the same Things, are to be reckon'd as if there were but one Thing in it: And if in any Sett there is any Part consisting of the same Things, reckon all that Part as it were but one Thing; then apply the Rule of *Problem* 5th.

Exa. Suppose 3 Setts of Things, one of 6 Things all different; another of 4, but which are all the same Things; another of 8 Things, whereof 5 are like, and 3 are like. To find the Compositions of 3 in these three Setts, we must reckon the 2d Sett as having but 1 Thing, and the 3d as having 2, and then the Number sought is $6 \times 1 \times 2 = 12$.

a	*h*	*m*
b	*h*	*m*
c	*h*	*m*
d	*h*	*m*
e		*n*
f		*n*
		n

The *Reason* of this Rule is obvious. So, in the preceding Example, we can join each Thing in the first Sett with *h*, and with *m*, or with *n*; nor have we any more Choice, since every *h*, *m*, and *n* are the same.

Problem 11th.

To find the *Compositions* of any Number in a greater number of Setts, Circumstances being as in the last *Problem*.

Rule. Reckon all the Things that are the same, or like, in any Sett, as but one Thing, and proceed by *Probl.* 6.

Problem 12th.

To find the *Compositions* of any Number N in a lesser Number n of Setts, Circumstances being as in *Probl.* 10.

Rule. Apply the first 3 Steps of the Rule of *Probl.* 7; and for the third Step (*i.e.* taking the Elections of the Parts of N out of the corresponding Setts) we must distinguish *thus*: 1°, If any Sett has all its Individuals like, there is but one Election.

But,

But, 2°, if they are neither all like, nor all different, find the Elections of the corresponding Part of *N* in that Sett by *Probl.* 8.

The *Reason* is obvious.

SCHOL. In the Matter of *Compositions* we have hitherto suppos'd, that there is no Thing in one Sett which is the same, or like to what is in any other Sett: But now, if we suppose it otherwise, *i. e.* that there are several Things the same, or like, in several Setts, this is a new Limitation, and more difficult than the former, and indeed is such as I have not found how to bring under any General Rules: Some particular Cases have their proper Rules, that will be easily discover'd when they occur, but what has been already done being the principal Part of this Doctrine, and the Foundation of what else relates to it, I shall not insist much further on these Things, only add one Problem more, *viz.*

PROBLEM 13th.

The Individuals of several Setts of Things being the same or like Things, *i. e.* the Things that are in any one Sett, (however like or unlike among themselves) being the same, and the same in Number, as are in all the other Setts; To find the different Compositions of a number of these Things equal to the number of Setts,

RULE 1°. If any of the Things is oftner than once in the same Sett, reckon as it were but once; Then, 2°, take the Arithmetical Progression 1, 2, 3, *&c.* to such a number of Terms as the number of different Things in each Sett; take the Series of their Sums continually from the beginning, the last of them is the number of Compositions sought in two Setts. Again; Take the Sums of their Series continually from the beginning, the last of them is the number of Compositions in three Setts. Go on thus taking the Sums of every succeeding Series of Sums, and the last Term is the Number for a number of Setts equal to the number of Series from the beginning of the Work.

Thus:

1 . 2 . 3 . 4
1 . 3 . 6 . 10
1 . 4 . 10 . 20
&c.

If there are 2 Setts, and 4 Things in each, the Compositions of 2 are 10; if there are 3 Setts, the Compositions of 3 are 20; and so on.

a	*a*	*a*	
b	*b*	*b*	&c.
c	*c*	*c*	
d	*d*	*d*	

DEMON. Suppose only 2 Setts; then the first Letter *a* of the second Sett may be join'd with each of the first Sett; but the second Letter *b* is not to be join'd with the *a* of the first Sett, because that coincides with the Combination of the *a* of the second, and *b* of the first Sett; and so the *b* of the second Sett is combin'd only with the *b* of the first Sett, and with all below. For the same Reason the *c* of the second Sett is join'd only with the *c* of the first Sett, and with all below, and so on thro' the other Letters. All which joinings, taking the whole Combinations of each Letter of the second Sett with all from the same Letter downwards in the first Sett, as one Term, makes an Arithmetical Series 1, 2, 3, *&c.* to a number of Terms equal to the number of Things in the Sett; and it's plain, that the Sum of them is the total Compositions of 2 in the two Setts. Again; For the same Reason the *a* of the third Sett will join with each Composition of 2 in the preceding two Setts; but the *b* must begin only to be join'd with *bb* of the preceding Compositions, and with none in which *a* is; *i. e.* it will be in as many Compositions of 2 as are in two Setts which have one Thing fewer, which must be the next lesser Term of the second Series: And the *c* of the third Sett will in like manner be join'd with *cc*, and all the Compositions of 2 in the preceding two Setts which have neither *b* nor *a*, (*i. e.* as many as the Compositions of 2 in two Setts which have two Things fewer than the last) which must be the next lesser Term of the second Series; and so on thro' all the Letters of the third Sett: So that the total Compositions of 3 in the three Setts is the Sum of all the Terms of the second Series. The same *Reason* is manifest, however many Setts there are.

ARITH-

ARITHMETICK.

BOOK VI.

The Application of the Doctrine of Proportion, *in the Common Affairs of Life and Commerce.*

IN *Books* I, and II, the Fundamental Operations of *Arithmetick* are fully explain'd in their Abstract Practice; and for the Application, all is done with respect to *Addition* and *Subtraction* that can be reduced to General Rules; also for the more simple Applications of *Multiplication* and *Division*: But the Great and Universal Rule for the application of *Multiplication* and *Division* is founded in the Doctrine of *Proportion Geometrical*, and contain'd chiefly in *Probl.* 1, *Chap.* 3, *Book* 4. whose Use, in the Common Affairs of Life and Commerce, is of the Greatest Importance, and for the explaining of which this Book is design'd.

CHAP. I.

The Rule of Three, Golden Rule, *or Rule of* Proportion.

DEFINITION.

THIS *Rule* (which is the Foundation of most of what follows) is the Application of *Probl.* 1, *Ch.* 3, *B.* 4, call'd *The Rule of Three* from having three Numbers given to find a fourth; but more properly *The Rule of Proportion*, because by it we find a 4th Number proportional to 3 given Numbers, (*i. e.* which has the same Ratio or Proportion to one of 3 given Numbers as another of them has to the remaining one). And because of the necessary and extensive Use of it, 'tis call'd *The Golden Rule*.

But;

But, to define it with regard to Applicate Numbers, or Numbers of particular and determinate Things, it is,—"The Rule by which we find a Number of any kind of "Thing (as Money, Weight, *&c.*) so proportion'd to a Given Number of the same "Things, as another Number of the same, or different Things, is to a third Number of "the last kind of Thing. For the four Numbers that are proportional must either be all apply'd to one kind of Things; or two of them must be of one kind, and the remaining two of another; because there can be no Proportion, and consequently no Comparison of Quantities, of different Species: As for example, of 3 Shillings and 4 Days; or of 6 Men and 4 Yards. But of this more fully afterwards.

All Questions that fall under this Rule may be distinguish'd into two kinds: The first contains these wherein 'tis simply and directly propos'd to find a 4th Proportional to 3 Given Numbers taken in a certain order: As, if it were propos'd to find a Sum of Money so proportion'd to 100 *l.* as 64 *l.* 10 *s.* is to 18 *l.* 6 *s.* 8 *d.* or as 40 *lb.* 8 ounces is to 6 hundred weight. The second kind contains all such Questions wherein we are left to discover, from the Nature and Circumstances of the Question, That a 4th Proportional is sought; and, consequently, how the State of the Proportion (or Comparison of the Terms) is to be made; which depends upon a clear understanding of the Nature of the Question, and of Proportion. And after the Given Terms are duly order'd, as the Proportion ought to run, what remains to be done is, to apply *Probl.* 1, *Ch.* 3, *B.* 4. But, because there is something further in the Application, and, to remove all Difficulty as much as possible, the whole Solution is reduced to the following General Rule, which contains what further Direction is necessary for solving such Questions wherein the State of the Proportion is given; in order to which, 'tis necessary to premise these Observations.

1°. In all Questions that fall under the following Rule there is a Supposition and a Demand: Two of the Given Numbers contain a Supposition, upon the Conditions whereof a Demand is made; to which the other Given Term belongs; and it is therefore said to raise the Question, because the Number sought has such a Connection with it as one of these in the Supposition has to the other. For example; If 3 Yards of Cloth cost 4 *l.* 10 *s.* (here is the Supposition) what are 7 Yards 3 quarters worth? (here is the Demand or Question rais'd upon 7 Yards 3 quarters, and the former Supposition.

2°. In the Question there will sometimes be a superfluous Term; *i. e.* which, tho' it makes a Circumstance in the Question, yet is not concern'd in the Proportion, because it is equally so in both the Supposition and Demand. This superfluous Term is always known by being twice mention'd, either directly, or by some Word that refers to it. *Exa.* If 3 Men spend 20 *l.* in 10 Days, how much at that rate will they spend in 25 Days? Here the 3 Men is a superfluous Term, the Proportion being among the other 3 given Terms, with the Number sought; so that any number of Men may be as well suppos'd as 3.

RULE. 1°. The superfluous Term (if there is one) being cast out, state the other 3 Terms, thus: Of the two Terms in the Supposition, one is like the Thing sought (*i. e.* of the same kind of Thing the same way apply'd); set that one in the 2d or middle place; the other Term of the Supposition set in the 1st place (or on the left hand of the middle); and the Term that raises the Question, or with which the Answer is connected, set in the 3d place (or on the right hand); and thus the Extremes are like other, and the middle Term like the Thing sought. Also the 1st and 2d Terms contain the Supposition, and the 3d raises the Question; so that the 3d and 4th have the same Dependance or Connection as the 1st and 2d. This done,

2°. Make all the 3 Terms simple Numbers of the lowest Denominations express'd; so that the Extremes be of one Name. Then,

3°. Repeat the Question from the Numbers thus stated and reduced, (arguing from the Supposition to the Demand) and observe whether the Number sought ought to be greater

greater or lesser than the middle Term (which the nature of the Question rightly conceiv'd will determine) and accordingly multiply the middle Term by the greater or lesser Extreme, and divide the Product by the other, the Quote is like the middle Term, and is the compleat Answer if there is no Remainder: But if there is, then,

4°. Reduce the Remainder to the Denomination next below that of the middle Term, and divide by the same Divisor, the Quote is another part of the Answer in this new Denomination. And if there is here also a Remainder, reduce it to the next Denomination, and then divide. Go on thus to the lowest Denomination; where, if there is a Remainder, it must be apply'd fractionwise to the Divisor: And thus you will have the compleat Answer in a Simple or Mix'd Number.

Observe, If any of the Dividends is less than the Divisor, reduce it to the next Denomination, and to the next again, and so on, till it be equal to the Divisor.

EXAMPLES.

Quest. 1. If 3 Yards of Cloth cost 8 *s.* what is the Value of 15 Yards? *Answ.* 40 *s.* or 2 *l.*

Work.

```
y.   s.   y.
3 — 18 — 15
     15
    ———
3)  120 (40 s.
```

Explanation.

3 yards and 8 *s.* contain the Supposition; and 8 *s.* is like the Thing sought; therefore 8 *s.* is the middle term, and 3 *y.* on the left: Then the Demand arises on 15 *y.* and therefore it is on the right. Again; from the nature of the Question it is plain, that 15 *y.* requires more than 3 *y* *i. e.* the Answer must be greater than the middle term; wherefore I multiply 8 *s.* by 15, the Product is 120 *s.* which divided by 3, quotes 40 *s.* without a Remainder; So 40 *s.* or 2 *l.* is the Number sought.

Qu. 2*d.* If 4 *lb.* of Sugar cost 2 *s.* 9 *d.* what is the Value of 18 *lb*? *Ans.* 12 *s.* 4 *d.* 2 *f.*

Work.

```
lb.  s.  d.   lb.
4 — 2 : 9 — 18
    12
    ——
    33 d.
    18
    ——
4) 594 d. ( 148 d.
Rem. 2
     4
    ——
4)  8 f  (2 f.
```

Explanation.

The Supposition is in 4 *lb.* and 2 *s.* 9 *d.* this being like the Thing sought, which is connected with 18 *lb.* wherefore the Terms are stated according to the Rule: then the middle Term being mix'd, I reduce it to *d*, and then I argue thus: If 4 *lb.* cost 33 *d.* 18 *lb.* must cost more; therefore I multiply 33 *d.* by 18, and divide the Product 594 *d.* by 4; the Quote is 148 *d*, and 2 remains, which I reduce to *f*, and divide the Product 8 *f* by 4 the Quote is 2 *f*; so the Answer is 148 *d*: 2 *f* or 12 *s.* 4 *d.* 2 *f.* because 148 *d.* is by reduction 12 *s.* 4 *d.*

Qu. 3*d.* What is the Price of 50 *lb.* weight of Tobacco, when 32 *lb*: 12 *oz.* cost 4 *l.* 10 *s*? *Ans.* 6 *l.* 17 *s.* 4 *d.* 3 *f.*

Work.

```
lb.  oz.   l.  s.    l.
32 : 12 — 4 : 10 — 50
16          20       16
———         ——       ——
524 oz. — 90 s. — 800 oz
            800
  524 )   72000 (137 s.
            524
            ———
           1960
           1572
           ————
            3880
            3668
            ————
Remainder   212
             12
            ————
  524)     2544 d (4 d.
           2096
Remainder  ————
            448
              4
           ————
  524)     1792 f.(3 f. 220/524
           1572
           ————
Rem.        220
           ————
```

Explanation.

The Supposition being contain'd in 32 *lb*: 12 *oz.* and 4 *l.* 10 *s.* which is like the thing sought, and the Question arising upon 50 *lb*, therefore the Terms are duly stated. Then, to make them all simple Numbers, and the Extremes like, I reduce both Extremes to ounces, and the middle Term to *sh.* Then, I say, if 524 *oz.* cost 90 *s.* 800 *oz.* must cost more: Wherefore I multiply 90 *s.* by 800, and divide

divide the Product 72000 by 524, the Quote is 137 *s.* and the Remainder is 212; which reduced to *d.* and divided, gives 4 *d.* with 448 remaining; and this reduced and divided, gives 3 *f.* and 220 remaining: So the Answer is 137 *ſh.* 4 *d*: $3\frac{220}{524}$ *f.* or 6 *l.* 17 *s.* 4 *d.* 3 *f.* &c.

Qu. 4*th.* What are 5 yards of Ribbon worth, whereof 63 yards and 2 quarters cost 5 *l*? *Anſ.* 7 *ſh.* 10 *d.* 1 *f.* $\frac{242}{254}$

Work.

yd. qr.	*l.*	*yd.*	
63 : 2 —	5 —	5	
4	1	4	
254 —	5 —	20 qr.	
	20		
254)	2000 *s.*	(7 *s.*	
	1778		
rem.	222		
	12		
254)	2664 *d.*	(10 *d.*	
	254		
rem.	124		
	4		
254)	496 *f*	(1 *f*	
	254		
rem.	242		

Explan.

The Terms stated, and reduced according to the Rule, I find the Answer ought to be greater than the middle term, therefore I multiply 5 *l.* by 20, but the Product is leſs than the Diviſor; and ſo I reduce it to *ſh.* which makes 2000; and this divided by 254, quotes 7 *ſh.* The reſt of the Work is plain.

Qu. 5*th.* What Time will 7 Men be boarded for 25 *l.* when 3 Men paid 25 *l.* for 6 Months?

Anſ. 2 Months 16 Days (reckoning 28 Days to a Month.)

Work.

Men.	*Mo.*	*Men.*
3 —	6 —	7
	3	
7)	18	(2 *Mo.*
	14	
rem.	4	
	28	
7)	112 *da.*	(16 *da.*

Explanation.

The 25 *l.* is a ſuperfluous Number; then the Suppoſition is in the 3 Men and 6 Months, and the Demand regards the 7 Men: The Terms being all ſimple, I argue thus: If 3 Men are boarded 6 Months (for 25 *l.* or any Sum) 7 Men will be boarded for the ſame a ſhorter Time: Therefore I multiply 6 Months by 3, and divide the Product 18 by 7; whereby I find the Anſwer 2 Months 15 Days.

Qu. 6*th.* If the Carriage of 3 hundred weight coſt 10 *ſh.* for 40 Miles, how much ought to be carried for the ſame Price 25 Miles and 3 quarters?

Anſ. 4 Cw : 2 qr : 17 *lb.* $\frac{97}{103}$

Work.

M.	*Cw.*	*M. qr.*
40 —	3 —	25 : 3
4		4
160 —	3 —	103
	160	
103)	480	(4 *Cw*
	412	
rem.	68	
	4	
103)	272	2 *qr.*
	206	
rem.	66	
	28	
103)	1848 *lb.*	(17 *lb.*
	103	
	818	
	721	
rem.	97	

Explanation.

The ſuperfluous Number is here 10 *s.* and the other 3 terms ſtated and reduced, I argue thus: If 3 Cw. is carried 160 qrs. of a Mile (for 10 *ſh.*) then a greater weight will be carried for the ſame Price 103 qrs. of a Mile: Therefore I multiply 3 by 160, and divide the Product 480 by 103, the Anſwer is, 4 Cw 2 qr : 17 *lb.*

Obſerve. The firſt Four Queſtions are what we call *The Rule of Three direct*; *i.e.* where the 3d Term, being greater or leſſer than the 1ſt, requires that the Anſwer be alſo greater or leſſer than the 2d Term. And the 2 laſt Queſtions are of the *Rule of Three Indirect* or *Reverſe*; where the 3d Term being greater or leſſer than the 1ſt, requires the 4th contrarily leſſer or greater than the 2d: But I have comprehended both in one General Rule. And from this Obſervation you learn how to know what Queſtions are of either kind.

In the following Collection of Queſtions I ſhall only ſtate the Proportion, ſet down the Anſwer, and leave the reſt of the Work to your

your own Exercise; and, by comparing the Answer which you find, with what is here, you'll know whether it is right.

Qu. 7th. What is the Value of 8 Chalder : 3 quarters, and 5 Bushels of Corn (*English* Measure) at the rate of 1 *l.* 15 *sh.* the Chalder?

Ans. 15.*l* : 11 *s* : 8 *d* 2 *f* $\frac{1}{2}$. State 1 Ch. — 1 *l* : 15 *s.* — 8 Ch. 3 qrs : 5 Bush.

Qu. 8th. When I bought 40 Gallons of Brandy for 6 *l* : 8 *sh* : 3 *d.* what is the Rate *per* Gallon? *G. l. s. d. G.*

Answer. 3 *s* : 2 *d* : $1\frac{9}{10}$ *f.* State 40 — 6 : 8 : 3 - 1.

Qu. 9th. In what Time will 13 Men finish a Work which 5 such Men could do in 3 Days : 8 Hours? *m. d. h. m.*

Answer. 1 da : 6 ho : 46 min : & $\frac{2}{13}$ State. 5 — 3 : 8 — 13.

Qu. 10th. If 3 Men are boarded 9 Months and 20 Days for 30 *l.* how many Men will the same Money pay for at that rate for 4 Months? *Mo. d. Men Mo.*

Answer. $7\frac{3}{12}$ or $\frac{1}{4}$ 9 : 20 — 3 — 4

Qu. 11th. 1 Man is boarded 3 Months for 5 *l.* what will he be owing if he remains at Board 8 Months and 7 Days? *mo. l. mo. da.*

Answer. 13 *l* : 15 *s.* State, 3 — 5 — 8 : 7

Qu. 12th. If I rode 60 Miles in 3 Days, when the Day was 14 Hours long (counting from Sun rising to setting) how long must the Day be, that I may ride 100 Miles in the same Time? *Mi. h. m.*

Answer. 23 ho : 20 min. State, 60 — 14 — 100

Qu. 13th. In what Time could I travel 50 Miles, the Day being 12 Hours long, at the rate of 50 Miles in 5 Days, when the Day is 16 Hours. *Ho. Da. Ho.*

Answer. 3 Days : 12 Hours. State, 12 — 5 — 16

Observe: If in this Question we put 5 Days : 7 Hours, instead of 5 Days, the State of the Question, according to the General Rule, is $\overset{ho.}{12} — \overset{da.\ h.}{5 : 7} — \overset{ho.}{16}$ But then take care that the middle Term be reduced to Hours by 12, this being the suppos'd Number of Hours in these Days: And this being done, there's no other Multiplication; and what remains to be done is, to divide the Hours produced by the Reduction, *viz.* 167 by 16, whereby they are reduced to Days of 16 Hours long, the Answer being 4 Days 3 Hours: So that the Question is in effect two Questions of Reduction, *viz.* 1st, of 5 d : 7 h. to Hours by 12, and these Hours again to Days by 16. Nor are the 4 Terms truly proportional; for their State, according to the Operation, is this, $\overset{ho.}{12} — \overset{da.\ ho.}{5 : 7} — \overset{ho.}{16} — \overset{da.\ ho.}{4 : 3}$. But to compare them as to Proportion, we must reduce the mix'd Number; *that is,* 5 Days 7 Hours, to Hours by 12; and 4 Days 3 Hours by 16; and so the 4 Terms are $\overset{ho.}{12} — \overset{ho.}{67} — \overset{ho.}{10} — \overset{ho.}{67}$; in which two Terms being equal, and the other two not so, the 4 cannot be proportional in any Order they can be taken.

The same will be also true tho' the middle Term be a simple Number, provided the 4th Term sought be not also a simple Number; but if 'tis, then there is Proportion. *Thus,* if for 16 we put 15, the Answer of *Qu. 13th* is simply 4 Days. And these Numbers are proportional Indirectly, *viz.* $\overset{h.}{12} : \overset{da.}{5} :: \overset{h.}{15} : \overset{d.}{4}$; or these directly, $\overset{h.}{15} : \overset{h.}{12} :: \overset{d.}{5} : \overset{d.}{4}$

But as the given Numbers are $\overset{h.}{12} — \overset{d.}{5} — \overset{h.}{16}$, the *Answer* is 3 *da* : 12 *ho.* and the 4 Terms are $\overset{h.}{12} — \overset{d.}{5} — \overset{h.}{16} — \overset{d.\ h.}{3 ; 12}$; and if we reduce the 2d and 4th Terms to one

Name (the one by 12, the other by 16) the 4 are $\overset{b.}{12} - \overset{b.}{60} - \overset{b.}{16} - \overset{b.}{60}$; which are proportional in no Order.

Qu. 14*th*. If a Piece of Cloth is 20 Yards in length, and $\frac{3}{4}$ in breadth, how broad is another Piece which is 12 Yards length, and contains as much Cloth as the other?

Answer. 1 Yard : 1 quarter. State, $\overset{yd.}{20} - \overset{qr.}{3} - \overset{yd.}{12}$

Qu. 15*th*. How much Shalloun of 1 Yard : 1 quarter breadth, will serve to line a Cloak of 5 Yards Cloth, 3 quarters broad?

Answer. 3 Yards State, $\overset{qr.}{3} - \overset{yd.}{5} - \overset{y.}{1} : \overset{q.}{1}$

Qu. 16*th*. If the Rate of Carriage is 1 Penny for 1 Pound weight carried 50 Miles, how far ought 1 Pound to be carried for 15 Shillings?

Answer. $37\frac{1}{2}$ Miles State, $\overset{d.}{1} - \overset{m.}{50} - \overset{sh.}{15}$

Qu. 17*th*. When Wheat is at 12 *sh.* *per* Bushel, the 6 *d.* Loaf of Bread weighs 1 *lb.* 4 *oz.* (*Troy* weight) what ought it to weigh, the Wheat being 9 *sh.* 6 *d.* the Bushel?

Answer. 16 *oz* : $14\frac{3}{7}$ *dw.* State, $\overset{s.}{12} - \overset{lb.}{1} : \overset{oz.}{4} - \overset{s.}{9} : \overset{d.}{6}$

Note, There are here two superfluous Numbers, *viz.* 1 Bushel, and 6 *d.*

Qu. 18*th*. What was the Price of Wheat when the Penny-Loaf of Bread weigh'd 8 Ounces; the Statute being, that it must weigh 10 Ounces, the Wheat at 12 *sh.* the Bushel?

Answer. 15 *s.* State, $\overset{oz.}{10} - \overset{s.}{12} - \overset{oz.}{8}$

Note, Here are two superfluous Numbers, *viz.* 1 Bushel and 1 Penny.

Qu. 19*th*. Three Pound weight of Bread costs 2 *sh.* 6 *d.* the Wheat at 14 *sh.* the Bushel; What is Wheat worth if I pay 2 *sh.* for the same weight of Bread?

Answer. 11 *s* : 2 *d* : $1\frac{3}{5}$ *f.* State, $\overset{s.}{2} : \overset{d.}{6} - \overset{s.}{14} - \overset{s.}{2}$

Qu. 20*th*. What is the Interest of 64 *l.* for 1 Year, the Rate of Interest being 5 *l.* 10 *s.* to 100 *l.* for 1 Year?

Answer. 3 *l* : 10 *s* : 4 *d* : $3\frac{1}{5}$ *f.* State, $\overset{l.}{100} - \overset{l.}{5} : \overset{s.}{10} - \overset{l.}{64}$

Qu. 21*st*. In what Time will 500 *l.* yield 40 *l.* Interest, when 86 *l.* does it in 4 Years 8 Months?

Answer. $20\frac{16}{25}$ Mon. State, $\overset{l.}{86} - \overset{y.}{4} : \overset{m.}{8} - \overset{l.}{500}$

Qu. 22*d*. At 6 *l.* *per Cent. per Annum*, what Principal Sum must be employ'd to yield 6 *l.* in 2 Year : 6 Months?

Answer. 45 *l* : 3 *s* : 2 *d* : $2\frac{16}{31}$ *f.* State, $\overset{y.}{1} - \overset{l.}{100} - \overset{y.}{2} : \overset{m.}{6}$

Qu. 23*d*. Of what Principal Sum did 20 *l.* Interest arise in 1 Year, at the rate of 5 *l.* *per Cent. per Annum?*

Answer, 400 *l.* State, $\overset{l.}{5} - \overset{l.}{100} - \overset{l.}{20}$.

Observe: In this, and *Qu.* 20*th*, and all *Questions* where the Terms represent the same kind of Things, there is in the nature of the *Question* some other Speciality, to distinguish them, which must be carefully observ'd, in order to make the 1st and 3d, 2d and 4th in all respects of one Application: So, in the *Questions* 20*th* and 23*d*; the Distinction is of Principal and Interest; in *Qu.* 14*th* and 15*th* it is Length and Breadth; and in *Qu.* 19*th* 'tis Price of Bread, and Price of Wheat.

DEMON-

Demonstration *of the preceding* Rule.

To this *Demonstration* I must premise these two things:

1°. When Numbers are consider'd complexly with the Things (or Quantities) they represent, or of which they are the Numbers (whence they are call'd *Applicate Numbers*) Then it's plain there can be no Comparison, or no Proportion, betwixt two such Numbers, unless they be of Things of the same kind, whose Quantities can in a proper sense be contain'd in one another. For example; There can be no Comparison or Proportion of 3 *sh.* to 4 yards, or 3 Men to 4 Days: But of any Quantity of Money to another; or of any kind of Weight or Measure to another; or of a Number of Men to Men, there is a proper Comparison: But *Observe*, that in this last Case the Number only is the Subject of Comparison.

2°. Of these Things wherein there is Subdivision with different Species and Denominations; Tho' two Numbers, either of the same or different Denominations, have a real Proportion, yet it is not the same as betwixt the two Numbers taken abstractly, or purely as Numbers, except when the Denominations are the same. Thus, 4 *lb.* and 7 *lb.* have a real Proportion, which is the same as that of the Numbers 4 and 7, taken abstractly; for, as 4 is $\frac{4}{7}$ of 7, so is 4 *lb* $\frac{4}{7}$ of 7 *lb.* But tho' 3 Ounces have also a real Proportion to 7 *lb*, 'tis not the Proportion of the Number 4 to 7, because 4 *oz.* are not $\frac{4}{7}$ of 7 *lb*; And the real Proportion of 4 *oz.* to 7 *lb*, reduced to that of abstract Numbers, is the Proportion of 4 to 112; *i. e.* 4 *oz.* to 112 *oz.* equal to 7 *lb.* *Universally*,

Of two Numbers, simple or mix'd, of the same kind of Things, (Money, Weight, Measure, *&c.*) the Proportion reduced to that of pure Numbers, is the Proportion of these two Numbers of the same Denomination to which the Given Numbers are respectively equal. *Exa.* The Proportion of 3 *sh.* to 8 *d.* is that of 36 to 8 (*viz.* 36 *d.* to 8 *d.*) Of 4 *l.* 6 *sh.* to 15 *sh.* 'tis 86 to 15, (*viz.* 86 *s.* to 15 *s.*) Of 9 *lb.* to 3 *lb.* 5 *oz.* 'tis 144 to 53 (*viz.* 144 *oz.* to 53 *oz.*) and so of others.

Follows the Demonstration,

1°. The three Given Numbers being stated according to the Rule, whereby the middle Term is like the Thing sought, and the Extremes like other, it follows from the first *Premise*, that the Extremes are the Terms to be compar'd, which contain the Proportion requir'd to be betwixt the middle Term and that sought: For, tho' the 1st and 2d Terms contain the Supposition upon which the Question arises, yet the given Proportion is originally betwixt the Extremes; and the Question express'd according to that Proportion is, to find a Number of Things like the middle Term, bearing the same Proportion to it as the 3d Term does to the 1st. Thus; In *Question 1st* the mix'd Proposition is, to find the Value of 15 yards of Cloth such, that 3 yards be worth 8 *sh.* But, the nature of the Question consider'd, it resolves plainly into this, *viz.* finding a quantity of Money so proportion'd to 8 *sh.* as 15 yards are to 3 yards; And therefore the Terms may be stated also thus: $\overset{y.}{3} - \overset{y.}{15} - \overset{s.}{8}$; whereby the 1st and 2d, 3d and 4th, are the compar'd Terms. But the other Way of stating them is more agreeable to the simple and obvious sense of the Question, and the Way of Reasoning with it upon which the Rule for multiplying and dividing is founded; which gives the true Answer according to the Proportion, as will be

shewn

shewn in the next Article: And I conclude *this* with observing, that as the Extremes contain the Proportion which the 4th ought to have to the 2d, so, if they are not of one Name, (by being mix'd Numbers, or otherways) they ought, by *Premise* 2d, to be reduced to one Name, and then the Proportion is reduced to pure Numbers. And, for the reduction of the middle Term, 'tis chiefly done for Convenience in the following *Multiplication* and *Division*; and not to produce an abstract Number; for its Application must always go along with it in the Product and Quote made by the Extremes (now become abstract Numbers). Thus you see the *Reason* of the 1st and 2d Articles of the Rule.

2°. The *Question* being resolv'd into this, *viz.* finding a Number like the middle Term, and in such Proportion to it as the 3d to the 1st, in some Cases (which is Direct Proportion) or as the 1st to the 3d in others; (which is Indirect, with regard to the Order in which the Terms stand, for this is all the Meaning of the distinction). And, the Extremes being now Abstract Numbers, it's manifest from *Probl.* 1, *Ch.* 3, *B.* 4, that the 4th is truly found by multiplying the middle Term by one Extreme, and dividing the Product by the other, according to the Rule: For the 3d Term being greater or lesser than the 1st, and the Question requiring the 4th Term also greater or lesser than the 2d, that Proportion is Direct, and the 3d Term is the Multiplier by the Rule, as it ought to be by *Probl.* 1, *Ch.* 3, *B.* 4. But if the 4th ought to be contrarily lesser or greater than the 2d, then the 1st Term is the Multiplier, by the Rule, as it ought to be, since the real Proportion is in the reverse order of the Terms. And here also *observe*, that tho' the Terms were stated in the plain Order of the Proportion; *i. e.* the 1st and 2d Terms made like other (as above shewn) yet the Operation would prove the very same; therefore the other Way is chosen for another Reason, already mention'd.

3°. The Reason for Reduction of the Remainders to lower Denominations is obvious.

Wherefore the *Whole Rule* is Compleatly demonstrated.

But, for their sake who have pass'd over all the Theory of *Proportion* (which will certainly hinder their being in any tolerable degree Masters of the Application; for, at least, a few of the more fundamental Notions and Theorems ought to be well understood, even for the sake of common Affairs) I shall here add another easy and simple *Demonstration* of that part of this Rule which depends upon *Probl.* 1, *Ch.* 3, *B.* 4. *Thus*,

Take the First Question stated, *viz.* If 3 yards cost 8 *s.* what will 15 yards cost? I suppose (1°) that it is ask'd, If 1 yard cost 8 *s.* what will 15 yards? Here it's plain the Answer is 15 times 8 *s.* or 120 *s.* Again, (2°) Let it be ask'd, If 3 yards cost 8 *s.* what will 1 yard? Here it's as plain, that the Answer is the 3d part of 8 *s.* Now, since 120 *s.* (or 15 times 8 *s*) is the Value of 15 yards, only upon Supposition that 1 yard cost 8 *s.* If, instead of this, one yard cost but the 3d part of 8 *s.* (as 'tis when 3 yards cost 8 *s.*) then it follows that 15 yards cost but the 3d part of 120 *s.* And so you see the Reason both of the Multiplication and Division; and the same way of reasoning will answer in all Cases of the *Rule of Three Direct* (*i. e.* where the 3d Term is Multiplier).

Again; Take the 5th Question, If 3 Men take 6 Months to do any thing in, how long will 7 Men take? 1° say, If 3 Men take 6 Months, what will 1 Man take? It's plain he must take 3 times as much Time, or 3 times 6 Months (= 18 Months). 2° say, If 1 Man take 18 Months, how much will 7 Men take? Here it's as plain they take but the 7th part of 18 Months; whence the Reason of both parts is manifest: And the same Reasoning will hold in all Questions of the *Indirect Rule of Three*, (*i. e.* where the 1st Term is the Multiplier).

OBSERVA-

OBSERVATIONS *relating to the Application of the preceding* RULE, *and of Simple* Multiplication *and* Division.

1. It has been already observed, and I shall repeat it, that all Simple Questions in *Multiplication* and *Division* are really Questions of Proportion, and if the Numbers are applicate, they are Questions of the *Rule of Three*. Thus, to multiply 3 by 4, is to find a 4th Proportional to 1 : 3 : : 4 ; and to divide 12 by 3 is to find a 4th Proportional to 3 : 12 : : 1. To Apply this; suppose 1 Yard costs 3 *s.* to find the Value of 4 Yards is finding a Number of Money proportioned to 3 *s.* as 4 Yards to 1 Yard, that is, as the Number 4 to 1 ; which is simply multiplying 3 *s.* by 4. Again, suppose 3 Yards cost 12 *s.* to find the Value of 1 Yard is finding a Number of Money proportioned to 12 *s.* as 1 Yard to 3 Yards, or the Number 1 to 3, which is simply dividing 12 *s.* by 3 to find a 3d part of it. And thus it is plainly in all Cases of the *Rule of Three* where the Extremes being of one Name, one of them is Unity. In all other Cases it is a mixt application of Multiplication and Division ; Where observe, that tho' the Multiplier and Divisor are, in the Question, applied to things of different kind from the number multiplied or divided, which in Simple Multiplication and Division I have fully shewn, in § 4, and 5. *Ch.* 7. *B.* 1. to be absurd, yet here that absurdity is removed by the extremes becoming, or being considered as abstract Numbers. Hence again,

2. We are further confirmed in the absurdity of all these Questions, proposed as Simple Questions of Multiplication, wherein both Terms are applicate, as the multiplying 3 *l.* 10 *s.* by 4 *l.* or by 4 *l* 6 *s.* and such like ; For if the Question belongs to Multiplication Simply, then it must resolve into a Question of the Rule of Three, wherein the Terms that contain the proportion (which in such Cases are the 1st and 2d) being of the same Name, the 1st Term or Divisor is Unity, which leaves the Solution upon the Multiplication. But the only Sense the Question can receive is, to find a Number like the 3d Term (or proposed Multiplicand) so proportioned to it as the 2d Term (or proposed Multiplier) reduced to its lowest Name, is to an Unit of that Name : It follows that the true *General Rule* for such Questions is, to reduce the proposed Multiplier to its lowest Name, and then multiply by it as an Abstract Number.

But if the Unit that regulates the Proportion is taken not of the lowest Name in the proposed Multiplier ; then, the Question is not in any sense Simple Multiplication ; for Division also is necessary to solve it. For *Exa.* If by multiplying 3 *l.* 10 *s.* by 4 *l.* 6 *s.* be meant finding a Number of Money so proportioned to 3 *l.* 10 *s.* as 4 *l.* 6 *s.* is to 1 *l.* then the Proportion is as 86 *s.* to 20 *s.* and the Solution is by multiplying 3 *l.* 10 *l.* by 86, and dividing the Product by 20, according to the *Rule of Three* ; and as many different Units as you can suppose for the 1st Term, so many different Questions, and consequently different Answers there must be. To sum up all ; if no qualification is expressed, then the solving of the Question by the *General Rule* last mentioned, is the only true Solution answering to the Notion of *Simple Multiplication :* And if any other Unit than of the lowest Name in the Multiplier is supposed, it is a Question of the Rule of Three, requiring both Multiplication and Division. And, at last I must observe, that the generality, who propose such Questions, expect the answer as if the Unit were of the highest Name in that kind to which the Multiplier belongs (tho' perhaps they are ignorant of the meaning of it) wherefore, to satisfy their Ignorance, state the proportion so, and work it accordingly. So the preceding Question stands thus; as 1 *l.* to 4 *l.* 6 *s.* so is 3 *l.* 10 *s.* to the Number sought ; And working by the Rule of Three, the prepared State and Answer of the Question is, as 20 *s.* to 86 *s.* so is (3 *l.* 10 *s.* or) 70 *s.* to 301 *s.* (15 *l.* 1 *s.*)

Several of our Writers on Arithmetick differ about the Solution of such Questions, without seeming to understand, at least not explaining the true meaning of them (tho' others

thers have declared the Abſurdity) and their Ignorance appears further, by ſuppoſing that the Multiplier ought always to be of the ſame kind of thing as the Multiplicand; but if we conſider the real meaning and import of ſuch Queſtions, it appears plainly that the propoſed Multiplier may be of any other kind of thing; ſo, if multiplying 3 *l.* 10 *s.* by 4 Yards 3 Qrs. can have no other Senſe than finding a number of Money proportioned to 3 *l.* 10 *s.* as 4 Yards 3 Qrs. is to 1 Qr. (or 1 Yard or 1 Nail, for we may ſuppoſe any of them) the Queſtion is as good Senſe as if the Multiplier were 4 *l.* 3 *s.* And in fact, ſuch Queſtions happen in all proportions of Money, and other things valued by Money.

Again, tho' they make no ſuch abſurd Queſtions in Diviſion as dividing one kind of thing by a quite different kind (as 18 *l.* by 3 Yards) yet theſe are equally reaſonable with theſe in Multiplication, when the neceſſary qualification is applied; That is, when the meaning is to find a number of one kind of thing ſo proportioned to another Number of the ſame kind, as an Unit of any other kind of thing is to any Number of this kind. For the Unit and that other Number being of one Name, they are as abſtract Numbers; and the Queſtion reſolves into a Simple Diviſion. So dividing 18 *l.* by 3 Yards in this Senſe is only dividing 18 *l.* by 3, as happens in this Queſtion. If 3 Yards coſt 18 *l.* what will 1 Yard coſt. But if the Unit is of another denomination, then the loweſt in the diviſor after reduction; both muſt be reduced, and the Solution takes in both Multiplication and Diviſion; as in this Queſtion, If 3 Qrs. coſt 18 *l.* what will 1 Yard? that is, If 3 Qrs. coſt 18 *l.* what will 4 Qrs?

3. *This Rule of Three* is the great Rule of Calculation in all kind of Affairs; but to give particular Directions for its Application in all the Variety of circumſtances, where proportions ariſe is impoſſible; for Queſtions may be leſs or more complex; comprehending various Queſtions of proportion connected in their circumſtances, either to bring out ſeveral Numbers required, or as ſo many neceſſary Steps towards the finding of one Number required; and beſides the proportions contained in a Queſtion, there may be other Operations of Addition and Subſtraction, Simple Multiplication or Diviſion, neceſſary either to make out the Terms of a Proportion, or after the Proportions are ſolved to find ſome Number ſought, or a Number to be further applied towards finding Numbers ſought; in ſhort to ſatisfy ſome condition of the Queſtion in the progreſs of the Work.

The manageing of ſuch Queſtions depends upon the Arithmetician's Judgment in diſtinguiſhing all the parts of the Queſtion, and knowing what each requires according to the true Senſe and Import of it, and of the ſeveral Operations of Arithmetick, and particularly of Proportion; of all which he muſt have a clear and ready Idea; and as there is no other general direction that can reach all Caſes, the only thing more that can be done to help one to acquire the neceſſary Capacity for all uſeful Queſtions, is to make the application particularly to ſuch Variety in all the common Subjects and Branches of Buſineſs, that who underſtands theſe may be ſuppoſed capable to do any other of the ſame, or any other uſeful kind. To this purpoſe are all the other common Rules that are brought in after the Rule of Three; which are applications of it chiefly; of which you have a large courſe in the following Chapters; and I ſhall add to this a few more complex Queſtions upon ſome of the former Heads, which come not ſo well under any of the following; but firſt I make this other general Obſervation.

4. If Fractions are among the given Numbers of a Queſtion of the *Rule of Three*; The procedure is in all reſpects the ſame, having due regard to the nature and operations of Fractions; for if the extremes are (or be made by reduction) Simple Numbers, Integral or Fractional, of the ſame denomination, and the middle Term a Simple Integer or Fraction; the Multiplication and Diviſion is to be performed by the Rules of Fractions, where Fractions are concerned: A few Examples are ſufficient to explain this.

Exa.

Exa. 1. If $\frac{3}{5}$ of a Yard cost 4*s.* $6\frac{2}{3}$ *d.* What is the Value of $24\frac{5}{7}$ Yard? State $\frac{3}{5}$ Yard — 4*s.* $6\frac{2}{3}$ *d.* — $24\frac{5}{7}$ Yard. By reduct. 'tis $\frac{3}{5}$ Yard — $\frac{164}{3}$ Yard — $\frac{173}{7}$ Yd. the Product of $\frac{164}{3}$ and $\frac{173}{7}$ is $\frac{28372}{21}$ which divided by $\frac{3}{5}$ the Quote comes out $\frac{141860}{63}$ *d.* equal to $2251\frac{47}{63}$ *d.* or 9*l.* 7*s.* 7*d.* $2\frac{62}{63}$ *f.*

Exa. 2. If $4\frac{5}{9}$ Ounces cost 8*s.* what cost $30\frac{3}{7}$ Pounds? the Extremes being reduced first to Simple Fractions, it is $\frac{41}{9}$ Ounces — 8*s.* — $\frac{213}{7}$ Pound, and the Extremes being again reduced to one Name, it is $\frac{41}{9}$ Ounces — 8*s.* — $\frac{3408}{7}$ Ounces, then 8 Multiply, by $\frac{3408}{7}$ produces $\frac{27264}{7}$, which divided by $\frac{41}{9}$ quotes $\frac{245376}{287}$ *s.* equal to $854\frac{278}{287}$ *s.* or 42*l*: 14*s*: 11*d.* $1\frac{29}{287}$ *f.*

Exa. 3. If 3 hundred weight: 2 qrs. $14\frac{3}{4}$ *lb.* cost 68*l*: 10*s.* What cost $1\frac{3}{8}$ hund. weight? First it is, $406\frac{3}{4}$ *lb.* — 1370 *s.* — $\frac{11}{8}$ Cw, then $\frac{1627}{4}$ *lb.* —1370 *s.*—$\frac{11}{8}$ Cw. then $\frac{1627}{4}$ *lb.* — 1370 *s.* — $\frac{1232}{8}$ *lb.* and in this State the Question is compleatly reduced and prepared, and the Answer found by multiplying 1370 *s.* by $\frac{1232}{8}$ and dividing the Product by $\frac{1627}{4}$

Observe. That tho' we cannot easily, in every Case, know which of the Extremes is greatest, unless they are reduced to one Denominator; yet without this it is easy to know which Extreme is the Multiplier; because, suppose that upon the right hand to be either the greater or lesser, that will determine which of them is the Multiplier. Yet after all, if you reduce the extremes to one Denominator; you'll have no more trouble, because the common Denominator may be neglected, and the Operation performed with the Numerators; since that Denominator would be a Multiplier both in the Numerator and Denominator of the Quote, as it comes out first in Fractional form; and therefore both being divided by it (or which is the same, neglect it in the Operation) the Quote will still be the same. So to Multiply by $\frac{5}{8}$ and Divide the Product by $\frac{3}{8}$ is the same as Multiplying by 5, and Dividing by 3.

Mixt QUESTIONS *for the* Rule of Three.

1. If 1 Yard of Cloth cost 15 *s.* at first buying, and upon 540 Yards, there was of Charges (as Packing, Carriage, &c.) 5 *l.* 10 *s.* What is the total cost? *Answer.* 410 *l.* 10 *s.* for if 1 cost 15 *s.* 540 cost 405 *l.* to which add 5 *l.* 10 *s.* the Sum is 410 *l.* 10 *s.*

2. A Gentleman has a Yearly Rent of 250 *l.* If he lays up Yearly 80 *l.* what has he to spend upon Living every Month? *Ans.* 14 *l.* 3 *s.* 4 *d.* For 250 — 80 = 170. then if 12 Months have 170 *l.* 1 Month has 14 *l*: 3 *s*: 4 *d.*

3. Having bought 40 Yards of Cloth at 8 *s. per* Yard; and 70 Yards at 12 *s.* What is the Value of both Pieces? *Ans.* 58 *l.* for if 1 Yard cost 8 *s.* 40 cost 16 *l.* then if 1 Yard cost 12 *s.* 70 cost 42 *l.* and 16 *l.* + 42 *l.* = 58 *l.*

4. I bestowed 80 *l.* upon 2 Pieces of Cloth; one of them at 13 *s. per* Yard, the other at 16 *s.* whose total Value is 48 *l.* What quantity was in each?

Take 48 from 80 remains 32, the total Value of the first Piece; then find the Quantities by the total Value, and Value of 1 Yard.

5. Five Persons were boarded together at 4 *l. per* Quarter; they paid at entry 1 *l.* 10 *s.* a Piece; and having continued 2 Year 3 Quarters; How much do they owe? The Solution is thus made 5 × 4 = 20 *l*, due for every Quarter; then if 1 Quarter cost 20 *l.* what 2 Year 3 quarters? From the Answer of this, Substract 5 times 1 *l.* 10 *s.* paid at entry, the remainder is the Answer.

6. Two Posts travel the one directly *North* and the other *South*, from the same Place; the one Travels 3 Miles for every 2 the other Travels, who Travels at the rate of 36 Miles a Day: How far are they asunder at the 3d Days end? 1°. 3 × 36 = 108 Miles the last Post has Travelled; then say, as 2 to 3 so is 108 to the Miles the other Travels; the Sum of the Miles travelled by both is the Answer: But if they had Travelled the same way, then it is the difference of what each Travels. Or suppose the last named is 30 Miles before the other; to find in what time that other comes up with him, say, As 2 to 3, so is 36 to the Miles the other Travels in a Day; the difference is what of the distance is taken away every Day; whence find the time in which the whole will be taken away.

7. In what time will two Persons do a Work which one of them can do in 8 Days, and the other in 5 Days? Find by the given proportion how much of the whole Work each could do in 1 Day: Add these Answers together; and say, if both together can do so much as that Sum in 1 Day; in what time will they do the supposed Work. In all which Operations that supposed Work is represented by 1.

8. Four Men Drink at Table 16 Pennyworth of Wine: How many Men, each of whom Drinks but half of what each of the other does, will 22 Penny worth serve? Four of these last Men will Drink but 8 Pence worth; therefore say, if 8 *d.* serves 4, what will 22 *d.*

2. Having bought 146 Yards of Cloth at a certain Price, the seller afterwards discompted 3 *l.* 10 *s. per. Cent*, and had in full Payment 248 *l.* What was the 1st Price of 1 Yard? Say, as 96 *l.* 10 *s.* is to 100 *l.* so is 248 *l.* to the first Price of the whole, By which find 1 Yard.

10. I laid out 10 *l.* upon a Parcel of Serges and Shalloons; the total Value of the Shalloons 60 *l.* and the total quantity of Serges 236 Yards; also for every 2 Yards of Serge, I had 3 of Shalloun: How much Shalloun was there? and, what was the Value of 1 Yard of each kind?

CHAP. II

Contractions in the Rule of Three; *called Rules of* Practice.

CASE 1. When a Question in the Rule of Three being duly stated, and the Extremes simple Numbers of one Name; whether the middle Term be Simple or mixt; if the Extreme which by the general Rule is the Divisor, be 1, and the middle Term an Aliquot part of some superior Species; then divide the other extreme by the Denominator of that aliquot part, the Quote is the Answer in that superior Species; and if there is a remainder reduce, and find its Value.

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TABLE *of the Aliquot Parts of* MONEY.

s.	d.	is equal so, one		part of a Pound.	d	f	is equal to		part of a Sh.	
:	6		40th		1 :			12th.		1 *f.* is ¼ part of 1 *d.*
:	10		24		1 :	2		8		
1 :			20		2 :			6		
1 :	8		12		3 :			4		
2 :			10		4 :			3d		
2 :	6		8		6 :			2d		
3 :	4		6							
4 :			5							
5 :			4							
6 :	8		3d							
10 :			2							

Examples.

1. What is the Price of 67 yards of Cloth at 5 *s. per* yard? *Answ.* 16 *l*: 15 *s.* found thus; The State of the Proportion is $\overset{yd.}{1} - \overset{s.}{5} - \overset{yd.}{67}$; And because the Divisor is 1, and the middle Term 5 *s.* which is a 4th part of 1 *l.* therefore I divide 67 by 4, the Quote is 16 *l.* and 3 remains, which reduced to Shillings, aud divided by 4, quotes 15 *s.*

The *Reason* of this Practice is obvious; for if 1 yard cost ¼ of 1 *l.* 67 yards must cost 67 4th parts, or, which is the same thing, the 4th part of 67 *l.*

As the Manner of Application, and Reason, in all other Examples, may be easily understood from this one, I shall only state a few, with their Answers.

2. The Value of 54 Stone weight, at 10 *s.* (½ of 1 *l.*) *per* Stone, is 27 *l.* (½ of 54.)

3. The Value of 353 Yards, at 3 *s* : 4 *d.* (⅙ of 1 *l.*) *per* Yard, is 58 *l* : 16 *s* : 8 *d.* (⅙ of 353 *l.*)

4. The Value of 365 *lb* weight, at 3 *d* (¼ of 1 *s.*) *per lb.* is 91 *s* : 3 *d*, or 4 *l* : 11 *s* : 3 *d.* (¼ of 365 *s.*)

5. If 48 Men do a piece of Work in 8 Hours, in what Time will 1 Man do the same? *Answ.* 16 Days, which is the 3d part of 48, because 8 Hours is the 3d part of 24 Hours or 1 Day.

CASE 2*d.* If the Price of an Unit is an even number of Shillings, multiply the other Extreme (of the same Name with the Unit) by the half of that Number; double the first Figure of the Product for Shillings, and the remaining Figures to the left are Pounds in the Answer.

Exa. 1. What is the Value of 324 Yards at 6 *s. per* Yard? *Ans.* 97 *l* : 4 *s.* thus; multiplying 324 by 3 (the ½ of 6) the Product is 972, which, according to the Rule, is 97 : 4 *s.* And *observe*, that it is very easy to set down the Shillings and Pounds separately, without writing first down the total Product and then separating them.

The *Reason* of this Practice is, that if we multiply the whole even Number of Shillings, the Product is the Answer in Shillings; which divided by 20, reduces it to Pounds, the remainder being Shillings: But if we multiply only the half of these Shillings, the Product is only the half of the Value in Shillings. Now suppose we multiply this Product by 2, to give the whole number of Shillings, and divide this last Product by 20 to reduce

them to Pounds; Then, because 20 is two times 10, it's plain that the Product made by the half of the given Price, being first multiply'd by 2, and this Product divided by 20, (or, which is the same thing, first by 2, and the Quote by 10) the last Quote will be the same as if that first Product were only divided by 10; because, to multiply by 2, and then divide the Product by 2, brings back the same Number that was multiply'd: Wherefore it's plain, that if the first Product is divided by 10, the Quote is the Answer in Pounds and 10th parts; and, because the Divisor is 10, therefore the Integral Quote, or Pounds, are express'd by the Dividend, excluding the first Figure on the right hand; and, because that Figure is the number of 10th parts, therefore double of it is the number of 20th parts, *i. e.* of Shillings; and thus every Part of the Rule is clear.

Observe; If the Price of an Unit consists of Pounds and Shillings, whose half reduced to Shillings is a Number by which we can easily multiply, so as to bring out the Product in one Line at the first Step, as we may if that half doth not exceed 29, then we may also use this Method.

Exa. What is the Price of 467 Yards at 1 *l*: 14 *s. per* Yard? Here 1 *l*: 14 *s.* is 34 *s.* whose half is 17, by which multiplying 467, according to the Rule, the Answer is 793 *l*: 18 *s.*

CASE 3*d.* If the middle Term is not an aliquot part of some superior Integer, (the Divisor being always 1) yet it may be equal to the Sum of several aliquot parts; and then if you divide by the Denominators of each of these separately, and add all the Quotes, the Sum is the Thing sought. The *Reason* is plain; but I must observe, that in most Cases where the middle Term is not an aliquot part, the common Rule by Reduction is easier Work.

Exa. 1. If 1 yard cost 15 *s.* what cost 49 yards? *Ans.* 36 *l*: 15 *s.* found thus; 15 *s.* is 10 *s.* and 5 *s.* *viz.* the $\frac{1}{2}$ and $\frac{1}{4}$ of 1 *l.* So I take the $\frac{1}{2}$ of 49 *l.* which is 24 *l.* 10 *s.* and $\frac{1}{4}$, which is 12 *l.* 5 *s.* whose Sum is 36 *l.* 15 *s.*

Exa. 2. If 1 yard cost 12 *s.* 6 *d.* what cost 268 yards? *Ans.* 167 *l.* 10 *s.* thus; 12 *s.* 6 *d.* is 10 *s.* and 2 *s.* 6 *d.* *viz.* $\frac{1}{2}$ and $\frac{1}{8}$ of 1 *l.* Then $\frac{1}{2}$ and $\frac{1}{8}$ of 268 *l.* make together 167 *l.* 10 *s.*

Exa. 3. If 1 yard cost 4 *s.* 3 *d.* what is the Value of 140 yards? *Answ.* 29 *l.* 15 *s.* thus; for 4 *s.* it is 28 (*viz.* the 5th part of 140) and for 3 *d.* it is 35 *s.* (or 1 *l.* 15 *s.*) the 4th part of 140 *s.* because 3 *d.* is $\frac{1}{4}$th of 1 *s.* and the Sum is 29 *l.* 15 *s.*

Observe; If the middle Term is equal to the same aliquot part repeated, *i. e.* if it's any simple Fraction (not an aliquot Part) you may either divide by the Denominator, and then multiply the Quote by the Numerator; or rather first multiply by the Numerator, and then divide by the Denominator.

Exa. 4. If 1 yard cost 13 *s.* 4 *d.* the Value of 56 is 37 *l.* : 6 *s.* : 8 *d.*; for 13 *s.* 4 *d.* make $\frac{2}{3}$ of 1 *l.* therefore I multiply 56 by 2, and divide the Product 112 by 3; the Quote is 37 *l.* 6 *s.* 8 *d.*

Again *Observe*, that if the middle Term is resolvable into Parts that are aliquot one of another, then it may often prove a convenient Practice, first to find the Answer for the greater Number, and then for the other which is its aliquot Part, by taking the like Part of the Answer for the former. *Exa.* If 1 yard cost 15 *s.* what 340 yards? Take the half of 340 for 10 *s.* and the half of this half for 5 *s.* your Sum is the Answer.

CASE 4. If the middle Term is so mix'd as to have in it any Number of the highest Species, first multiply that Number, and then the other Parts by some of the former Cases, if possible; and if this cannot be done, or not without much Operation, then we must take the common Method by Reduction.

734			
4			
2936 *l* :	*s* :	*d.*	
244 :	13 :	4	
3180 :	13 :	4	

Exa. 1. If 1 yard cost 4 *l.* 6 *s.* 8 *d.* what cost 734 yards? *Answ.* 3180 *l.* 13 *s.* 4 *d.* For 4 *l.* multiply'd by 734, produces 2936 *l.* and for 6 *s.* 8 *d.* I take $\frac{1}{3}$ of 734, which is 244 *l.* 13 *s.* 4 *d.* and the Sum of both is 3180 *l.* 13 *s.* 4 *d.*

28 by 5 is =	140 *l.*	*s.*	*d.*
For 6 *s.*	8 :	8	
For 10 *d.*	1 :	3 :	4
Sum,	149 :	11 :	4

Exa. 2. If 1 yard cost 5 *l.* 6 *s.* 10 *d.* what cost 28 yards? *Ans.* 149 *l.* 11 *s.* 4 *d.* thus; For 5 *l.* it is 140 *l.* for 6 *s.* I use *Case 2d*, and for 10 *d.* I take the 24th part of 28, by *Case 1st.*

For 1 *l.* =	38 :	*s.* :	*d.*
For 6 *l.* 8 *d*—	12 :	13 :	4
For 2 *l.* 6 *d*—	4 :	15 :	
Sum,	55 :	08 :	4

Exa. 3. If 1 yard cost 1 *l.* 9 *s.* 2 *d.* what cost 38 yards? *Answ.* 55 *l* 8 *s.* 4 *d.* thus; For the 1 *l.* 'tis 38, for 9 *s.* 2 *d.* which is equal to 6 *s.* 8 *d.* ($\frac{1}{3}$ of a Pound) and 2 *s.* 6 *d.* ($\frac{1}{8}$ of a Pound) I take $\frac{1}{3}$ and $\frac{1}{8}$ of 38 *l.* as in the Margin.

Suppose the Price of 1 yard 3 *l* : 7 *s* : 9 *d.* then no Method by aliquot parts is so easy as the common Method by Reduction: And I shall here *observe*, that as there may be different Ways of doing the same Question by aliquot parts, so the chusing of the best Way depends upon Experience. *Exa.* If 1 cost 15 *s* : 8 *d.* it may be done by considering 15 *s.* as $\frac{3}{4}$ of a *l.* or as $\frac{1}{2}$ & $\frac{1}{4}$; and 8 *d* as $\frac{2}{3}$ of a *s.* But more easily by resolving 15 *s* : 8 *d.* into 14 & 1 *s* : 8 *d.* Then, for the 14 *s.* work by *Case 2d*, and for 1 *s* : 8 *d.* by *Case 1st.* Also in *Exa.* 3 *d* we may work first by 1 *l* : 8 *s.* by *Case* 1.; then for 1 *s.* and for 2 *d*; but the former Method is easier, after you perceive that 9 *s* : 2 *d.* is equal to 6 *s.* : 8 *d.* and 2 *s* : 6 *d.*

CASE 5. If the Extreme which is the Multiplier is an aliquot part, or the Sum of certain aliquot parts, of the Unit which is Divisor, then take by Division such part or parts of the middle Term (whether this be a simple or mix'd Number): And if the Multiplier has also some Number of the same Species with the Unit, you must work for that Number separately by some of the former Cases, or the common Rule, then add all the Parts of the Answer.

Exa. 1. If 1 Pound weight cost 32 *l.* what cost 4 Ounces? *Ans.* 8 *l.* *viz.* $\frac{1}{4}$ of 32 *l.* because 4 *oz.* are $\frac{1}{4}$ of 1 *lb.*

Exa. 2. If 1 yard cost 3 *l* : 10 *s.* what cost 2 qr : 1 nail? *Ans.* 1 *l* : 19 *s* : 4 *d* : 2 *f.* thus; For 2 qr. I take $\frac{1}{2}$ of 3 *l* : 10 *s.* *viz.* 1 : 15, and for 1 nail I take $\frac{1}{16}$, which is 4 *s* : 4 *d* : 2 *f.* Total, 1 *l* : 19 *s* : 4 *d* : 2 *f.*

Exa. 3. If 1 *l.* buy 3 *Ct* weight : 1 *qr* : 7 *lb*, how much will 28 *l* : 5 *s* : 6 *d.* buy?

Ct.	*qr.*	*lb.*		*Ct*	*qr.*	*lb.*
3 :	1 :	7	4	3 :	1 :	7
28			10	3 :	8 :	12 *oz.*
84					9 :	4$\frac{4}{10}$
7				3 :	18 :	00$\frac{4}{10}$
1 :	03					
92 :	03					

Answ. 93 *Ct* : 2 *qr* : 18 *lb* : $\frac{4}{10}$ *oz.* which I find thus; 1°. For the 28 *l.* I multiply 3 *Ct* by it, which gives 84 *Ct*; then for 1 *qr.* I take $\frac{1}{4}$ of 28, is 7 *Ct*; and for 7 *lb.* I take $\frac{1}{16}$ of 28, is 1 *Ct* : 3 *qr*, or, which is the same, I take $\frac{1}{4}$ of 7 *Ct*, because 7 *lb* is $\frac{1}{4}$ of 1 *qr.* So the Total for 28 *l.* is 92 *Ct* : 3 *qr.* 2°. For 5 *s.* which is $\frac{1}{4}$ of 1 *l.* I take $\frac{1}{4}$ of 3 *Ct* : 1 *qr* : 7 *lb*, it is 3 *qr* : 8 *lb* : : 12 *oz.* and for 6 *d*, which is $\frac{1}{10}$ of 5 *s*, I take $\frac{1}{10}$ of 3 *qr* : 8 *lb* : 12 *oz*, it is 9 *lb* : 4$\frac{4}{10}$ *oz.* So the total for 5 *s* : 6 *d.* is 3 *qr* : 18 *lb* : $\frac{4}{10}$ *oz.* And, to this adding 92 *Ct* : 3 *qr*, the Sum is 93 *Ct* : 2 *qr* : 18 *lb* : $\frac{4}{10}$ *oz.*

Observe:

Observe: If the Multiplier and the middle Term are both of the same kind of Things, then we may consider either as the Multiplier, as shall be most convenient for the Operation.

Exa. 4. If 1 *l.* gain 4 *s* : 6 *d.* how much is thereby gain'd upon 34 *l* : 10 *s*? *Answ.* 7 *l* : 15 *s* : 3 *d.* Which is found either of two Ways, *viz.* 1°. Multiplying 4 *s* : 6 *d.* by 34 *l* : 10 *s.* thus; 4 *s.* by 34, makes 6 *l* : 16 *s.* and 6 *d* by 34 makes 17 *s*; then 4 *s* : 6 *d.* by 10 *d.* or $\frac{1}{2}$, makes 2 *s* : 3 *d*; and the Total is 7 *l* : 10 *s* : 3 *d.* Or, 2°, Multiplying 34 *l* : 10 *s.* by 4 *s* : 6 *d*, thus; 4 *s* : 6 *d.* is 2 *s* and 2 *s* : 6 *d*; Therefore I multiply 34 *l* : 10 *s.* by 2 *s.* or $\frac{1}{10}$, the Product is 3 *l* : 09 *s.* Then, by 2 *s* : 6 *d.* or $\frac{1}{8}$, it is 4 *l* : 6 *s* : 3 *d*; And the Total is, as before, 7 *l* : 15 *s* : 3 *d.*

	the Product of
4 *s* : 6 *d.*	
6 : 10 *s.*	4 *s.* by 34
17	6 *d.* by 34
2 : 3	4 *s* : 6 *d.* by $\frac{1}{2}$
7 : 15 : 3	

	the Product of 34 *l* : 10 *s.* by
34 *l* : 10 *s.*	
3 : 09	2 *s.* or $\frac{1}{10}$
3 : 06 : 3	2 *s* : 6 *d.* or $\frac{1}{8}$
7 : 15 : 3	

These are the *Chief* and *Fundamental* Practices by *Aliquot Parts*, which whoever understands will easily find many particular Abridgments depending upon the same Principles: But what I have done is sufficient here; Judgment and Experience will supply the rest better than a confus'd heap of *Particular Rules*.

CASE 6. When a Question of the *Rule of Three* is duly stated and reduced, according to the General Rule, 'twill often happen that you can easily discover a Number which will exactly divide the Extreme which is the Divisor, and some one of the other two Terms; substitute the Quotes of these Divisions in place of the Numbers divided, and work with them instead of these others; by this Means you'll have the Divisor and another Term reduced to smaller Numbers, and sometimes one of them will become 1, which leaves no more Operation but a simple Multiplication or Division.

Exa. 1. If 7 yards cost 56 *l.* what cost 35 yards? The Question stated is, 7 yards — — 56 *l.* — 35 yards; where it's easily perceiv'd that 7 divides both the Extremes, and the Quotes are 1 & 5: So that this Question 1 *yd* — 36 *l* — 5 *yd* will have the same Answer as the former, and is found simply by multiplying 56 *l.* by 5, which makes 280 *l.*

Exa. 2. If 250 *l.* buy 548 yards, what will 5 *l.* buy? The Extremes being both divided by 5, the Quotes are 50 and 1, and the Question will have the same Answer as this, 50 *l* — 548 *yds* — 1 *l.* which is solv'd by dividing 548 *yds* by 50; the Quote is 10 *yds* : 3 *qrs* : 3 *nai* : and $\frac{1}{50}$.

Exa. 3. If 27 yards cost 45 *l*, what cost 63 yards? Here the Extremes 27 & 63 being divided by 9, the Quotes are 3 & 7; and so the Question has the same Answer as this, 3 *yds* — 45 *l* — 7 *yds.* Again; 3 & 45 being both divided by 3, the Quotes are 1 & 15; and so the Question is reduced to this, 1 *yd* — 15 *l* — 7 *yds*, and the Answer 7 times 15, or 105 *l.*

You'll in many Questions discover at sight, or with a small attention, the Divisors that make this Abridgment, and also the Quotes; and in such Cases only is this Practice of any Value.

The *Reason* of this Practice is plain; for the two Numbers, equally divided, contain betwixt 'em the Proportion that ought to be betwixt the other given Number and that sought: But if two Numbers are equally divided, the Quotes (which are like aliquot Parts of them) are in the same Proportion.

From

From the same Principle it follows, that if the Divisor, and any of the other two Terms, are Fractions; or one of them a Fraction, and the other a whole Number; if these two Terms are reduced to Fractions having the same Denominator, you may neglect the common Denominator, and work with the Numerators; because Fractions, having a common Denominator, are in the same Proportion to one another as their Numerator.

Exa. If $\frac{3}{8}$ of a Yard cost 16 *l.* what cost $\frac{5}{8}$ of a Yard? This will have the same Answer as this, 3 *yds* — 16 *l* — 5 *yds.*

Exa. 2. If $4\frac{3}{5}$ Ounces cost 28 *l.* what cost 18 *lb.* or 4608 Ounces? The first Term reduced is $\frac{23}{5}$, and you may turn either 28 or 4608 into the form of a Fraction whose Denominator is 5, and then work with the Numerators; and so it will be either 23 *oz.*—140 *l* — 4608 *oz.* or 23 *oz* — 28 *l* — 23040 *oz.* The Answer is, $28048\frac{16}{23}$ *l.* In many Cases this may be useful.

CHAP. III.

The Rule of Five.

THIS *Rule* is so call'd from having 5 Numbers given to find a 6th; of which 5 Given Numbers, 3 are conjoin'd in the form of a Supposition, and upon that a Question is rais'd from the other 2, which with the Number sought are respectively like in their Application to the former 3, and have the same Connection of Sense; by which 'tis easy to know, at sight, a Question belonging to this Rule. Again; All Questions of this Rule are such as include two Questions of the Rule of *Three*, so dependent upon one another, that the Answer of the first being made the middle Term of the second, the Answers of both have the same Signification, and the last is the final Answer of the Question. And *Observe*, that tho' many Questions include two Questions of the Rule of *Three*, yet they have not the Conditions here describ'd (of which you'll meet with Examples afterwards); and, for the Solution of such as have, take this:

Rule.

1°. Of the 3 Terms of Supposition set that one first down which is like the Thing sought; towards the left hand of it set down the other two (it's no matter in what Order); then set the two Terms that raise the Question towards the right hand of the former 3, in such Order that of the 5 given Terms the 1st (counting from the left to the right) be like the 4th, and the 2d like the 5th: Then,

2°. Take the 1st, 3d, and 4th Terms in Order, of which make a Question of the Rule of *Three* (assuming the 2d Term for a superfluous one, to compleat the Sense) and find the Answer: Then,

3°. Take the 2d (of the 5) the Number now found, and the 5th (of the 5) in Order, and of them make another Question of the Rule of *Three*, (assuming the 4th Term of the 5 for a superfluous Number) the Answer of this is the final Answer sought.

QUESTIONS

QUESTIONS.

Qu. 1*st.* If the Carriage of 25 Stone weight, for 16 Miles, cost 15 *l*: 10 *s*. what will 40 Stone cost for 9 Miles?

Answ. 13 *l*: 19 *s*.

The State of the 5 Terms is,

St.	*Miles*	*l.*	*s.*	*St.*	*Miles.*
25 —	16 —	15 :	10 —	40 —	9

The first Question of the *Rule of Three* is,

St.	*l.*	*s.*	*St.*
25 —	15 :	10 —	40

by Reduction,

St.	*s.*	*St.*
25 —	310 —	40

```
         40  s.
25|12400 (496.
  100.
  ----
   240
   225
   ---
    150
    150.
```

2d Question.

m.	*s.*	*m.*
16 —	496 —	9

```
        9
      ----
16|4464 (279 or
   32    13l:19s
   ---
   126
   112
   ---
    144
    144
```

From the Sense and Connection of these two Questions it's plain, that we have the Answer of the Question propos'd; for, by the first we find what 40 Stone costs for 16 Miles, when 25 Stone cost 10 *s*. for the same Way. Then, by the second, we find what 40 Stone must cost for 9 Miles, when they cost 496 *s*. (the Answer of the preceding) for 16 Miles.

Qu. 2*d.* What Weight must be carried 12 Miles for 5 *l* : 4 *s*. when 18 Stone 10 *lb.* cost 15 *s*. for 7 Miles?

Answ. 15 *st* : 5 *lb* : 3 *oz* : $14\frac{1}{2}$ *dr*.

State of the 5 Terms.

s.	*m.*	*St.*	*lb.*	*l.*	*s.*	*m.*
15 —	7 —	18 :	10 —	5 :	4 —	12

1st Operation.

s.	*st.*	*lb.*	*l.*	*s.*
15 —	18 :	10 —	5 :	4

by Reduction,

s.	*lb.*	*s.*
15 —	298 —	104

```
      704      lb.
     -----
15|30992  (2066
  |20
   ---
   099
    90
    --
     92
     90
     --
     2 remainder
     16
     --
15|32 (2oz.
   30
   2 rem.
   16
   --
15|32 (2dr 2/15
   30
   2 remain.
```

2d Operation.

m.	*lb.*	*oz.*	*dr.*	*m.*
7—	2066 :	2 :	2 —	12

by Reduction,

m.	*dr.*	*m.*
7 —	528930 —	12

```
          7
      -------
12|3702510
```

Quote, $308542\frac{6}{12}$ *dr*

which being reduced, is equal to

st.	*lb.*	*oz.*	*dr.*
75 :	5 :	3 :	$14\frac{6}{12}$

Here, by the 1st Operation, is found how much must be carried 7 Miles for 5 *l* : 4 *s*. Then, by the 2d Operation, how much must be carried 12 Miles for 5 *l* : 4 *s*.

As to the Work of these two Questions, I have done it at length, according to the most General Rules; but such as understand the Contractions, already explain'd in *Multiplication* and *Division*, and the *Rule of Three*, may do them easier thus:

In the 1st Question, the 1st Operation reduces at sight to this 5 *st* — 15 *l* : 10 *s* — 8 *st*. by dividing 25 and 40, both by 5; then we may easily multiply 15 *l* : 10 *s* by 8, without reducing it; the Product is 124 *l*. which divided by 5, quotes 24 *l* : 16 *s*. For the 2d Operation it stands thus, 16 *m* — 24 *l* : 16 *s* — 9 *m*. And here the middle Term is easily multiply'd by 9 without Reduction; the Product is 223 *l* : 4 *s*, which divided by 16, (or first by 2, and the Quote by 8) the Answer is 13 *l* : 19 *s*.

In the 2d Question, the 2d Operation may be done easier; for, without reducing the middle Term, it may be easily multiply'd by 7; the Product is, 14462 *lb* : 14 *oz* : 14 *dr*; which

which divided by 12 (or 1st by 2, and this Quote 7231*lb* : 7*oz* : 7*dr.* by 6, the Quote is 3*oz.* 14*dr.* which is 75*st.* 5*lb.* 3*oz.* 14*dr.* as before.

Observe. That as the first Operation ought to be carried down to the lowest Species, while there is any remainder; so the Fraction which that remainder makes ought to be taken into the second Operation, else the final Answer will be thereby deficient, either in some Integral part, or in the Fractional part of the lowest Species. So in the preceding Question 2d, the $\frac{1}{17}$ of a drop in the Answer of the first Operation, being neglected in the 2d, makes the Answer of this deficient, tho' it is only by a small Fraction of a drop; yet in other Cases the loss may be more considerable, and therefore it ought never to be neglected: But because the taking in this Fraction into the 2d Operation will often make the Work tedious and hard, for them that are not familiar with the practice of Fractions, I shall give you another Rule, whereby the Answer is found by one Division; and because this Rule depends upon the preceding, you must have a clear understanding of that, in order to be master of this; and therefore I would have you first apply the preceding Rule to all the following Questions, and then apply the other Rule which is this.

How to solve Questions of the Rule of Five *by one Division.*

RULE 1°. State the 5 given Terms as before directed, and then make the corresponding Terms (*viz.* 1st and 4th, 2d and 5th) simple Numbers of one Name, and the middle Term a simple Number if its mixt.

2°. Form a Question of the Rule of Three with the 1st, 2d and 4th (of the 5) as before; and mark which of the extremes (1st or 4th) would be the Divisor: Again, form a Question of the Rule of Three with the 2d, 3d and 5th (of the 5) and mark which would be the Divisor.

3°. Let these two Terms, which you find would be the Divisors in these simple Questions of the Rule of Three, be multiplied together, and of the Product make a Divisor; and for a Dividend, take the continual Product of the other three Terms (of the 5) and this Division being finished (in which the Quote is like the middle Term) gives the final Answer of the Question proposed.

Observe. If the Numbers by which the middle Term is multiplied, are such that it may be easily multiplied without reduction when it's a mixt Number, then it's better not to reduce.

Quest. 1. Done by one Operation. The Question being stated and reduced according as the 1st and 2d Articles require, I make a Question of the Rule of Three with the 1st, 3d and 4th Terms, and find that 25 is the Divisor: For if 25*st.* cost 310*s.* (for 16 Miles) 40*st.* will cost more (for the same 16 Miles) therefore 40 is the Multiplier, and 25 the Divisor. Again, I make a Question upon the 2d, 3d, and 5th, and find 16 the Divisor; for if 16 Miles carriage (of 40*st.*) cost 310*s.* then 9 Miles must cost less; and so 9 is Multiier, and 16 Divisor.

St. *M.* *L.* *Sh.* *St.* *M.*
25 — 16 — 15 : 10 — 40 — 9.
Or thus
St. *M.* *St.* *St.* *M.*
25 — 16 — 310 — 40 — 9

St. *M.* *Sh.* *St.* *M.*
25 — 16 — 310 — 40 — 9
16 40
400 12400
9
400)111600)
Quote 279*sh.* or 13*l.* 19*s.*

The rest of the Work is manifest, as in the Margin; and because the 1st single Operation had no remainder in the lowest Species, the final Answer is the same by both Rules.

Observe

or without reducing the middle Term; thus,

l.	*s.*
15 :	10
	40

1620 :
9

400)5580(13 *l* 19 *s.*

Observe that I assume the middle Term for the Answer of the 1st single Question, because its no matter what be supposed in order to discover the Divisor.

Question 2d. By one Operation.

Sh.	*M.*	*St.*	*lb.*	*L.*	*Sh.*	*M.*
15 —	7 —	18 :	10 —	5 :	4 —	12

or

S.	*M.*	*lb.*	*S.*	*M.*
15 —	7. —	298	— 104 —	12
12		7		
180		2086		
		104		

180)216944(1205 *lb.*
18
36 or 75 *St.* : 5 *lb.*
36
094
90
44 remain.
16
180)704(3 *z.*
54
164 remains.
16
180)2624)14 $\frac{104}{180}$ *dr.*
180
824
720
104 remain

Here I say, 1st, if 15 *Sh.* pay for (7 Miles carriage of) 298 *lb*, then 104 pays for more; and so 104 is the Multiplier, and 15 the Divisor. Again if 298 *lb.* was carried 7 Miles for 104 *l.* a less weight must be taken 12 Miles (for the same Price) so 7 is the Multiplier, and 12 the Divisor: The rest of the Work is plain, as in the Margin. And here the final Answer differs from that found by the preceding Method (in which the Fraction of the 1st part was neglected) only in the Fraction of the lowest Species, which is here a little greater than the other.

The following Questions I leave wholly to your exercise, and only set down the Answers, as they are found by both the preceding Rules, that you may compare them with your own Answers. And observe, that in doing by two Operations I have always neglected the Fraction in the lowest Species of the 1st Operation.

Quest. 3. How far ought 3 *Ct.* and 2 qr. to be carried for 14 *s.* at the rate of 2 *l.* 10 *s.* for 14 *Ct.* carried 18 Miles? *Answer*, 20 Miles $\frac{4}{25}$ of a Mile, by both the Rules.

Quest. 4. If 246 *l.* board 9 Men 18 Months, how long will 48 *l.* board 5 Men? *Answer*, 6 Months 8 Days and $\frac{2}{5}$, by two Operations; and 6 Month 9 Days $\frac{18}{1230}$ or $\frac{3}{205}$ by one Operation. Observe, I have taken 28 Days to 1 Month.

Quest. 5. How much will pay 8 Months board of 3 Men, when 24 *l.* 5 *s.* paid for 2 Year 4 Months of 7 Men? *Answer*, 2 *l.* 15 *s.* 5 *d.* $\frac{2}{30}$ or $\frac{1}{15}$ by two Operations: And 2 *l.* 15 *s.* 5 *d.* $\frac{3}{21}$ or $\frac{1}{7}$ by one Operation.

Quest. 7. How many Men will cut down 7 Acres of Wheat in 4 Days, when 6 Men cut down 12 Acres in 8 Days and 4 Hours? *Answer.* 4 Men with a Fraction equal to $\frac{9}{10}$ by two Operations; and by one Operation it is 5 Men with a Fraction equal to $\frac{43}{60}$.

Quest. 7. If I get 8 *oz.* weight of Bread for 6 *d.* the Wheat at 15 *s.* *per* Boll; what ought the Boll of Wheat to be, that I may get 12 *oz.* of Bread for 4 *d.* *Answer.* 6 *s.* 8 *d.* by both Methods.

Quest. 8.

Qu. 8. When Wheat is at 12 *s.* 10 *d. per* Boll, 7 Ounces of Bread cost 5 *d.* how much ought to be got for 8 *d*, the Wheat being 15 *s* ? *Answ.* 9 *oz* : 11 *dw* : 14 *gr.* $\frac{2}{5}$, by two Operations; and 9 *oz* : 11 *dw* : 14 *gr.* $\frac{16}{45}$ by one Operation.

Qu. 9. What ought to be the Price of 4 *lb* : 10 *oz.* of Bread, the Wheat being 16 *s* : 5 *d.* the Boll, supposing that when the Wheat is at 12 *s.* I get 8 *oz.* for 4 *d.*? *Ans.* 3 *s* : 2 $\frac{1}{2}$ *d.* by two Operations; but by one Operation 'tis 3 *s* : 3 *d* : 2 $\frac{800}{1152}$ *f.*

Qu. 10. If 100 *l.* Principal Sum give 5 *l* : 10 *s.* in 1 Year, what is the Interest of 72 *l* for 5 Year 8 Months? *Answ.* By two Operations, 9 *l* : 4 *s* : 9 *d* : 1 *f* $\frac{60}{100}$, or $\frac{3}{5}$. And by one Operation, 9 *l* : 4 *s* : 9 *d* : 2 *f* $\frac{2}{5}$.

Observe, I have in this *Question* taken 12 Months to 1 Year.

Qu. 11. At the rate of 6 *l. per Cent. per Ann.* what Principal Sum will raise 48 *l.* in 2 Year 4 Months (supposing 12 Months to a Year)? *Answ.* By both Methods is 342 *l* : 17 *s* : 1 *d* : 2 *f* $\frac{6}{7}$.

Qu. 12. In what Time will 146 *l* : 10 *s.* Principal Sum raise 50 *l.* of Interest at the rate of 5 *per Cent. per Annum*? *Answ.* 6 Year : 10 Months : 16 Days, by two Operations; and by one it is 6 Year : 10 Months : 20 Days $\frac{188}{293}$.

Observe, I have here reckon'd 13 Months to a Year, and 28 Days to a Month; but, in Calculations of Interest, the most exact Way is, to take in no Denominations, but Years and Days (365 Days to 1 Year) and let the Time of a Question which is less than 1 Year, or that Part of it which exceeds a certain number of Years, be reckon'd in Days.

The *Reason* of this *Rule*, by one Division, will be easily understood by one Example. Thus,

Suppose 40 *l.* pay 7 Months board of 6 Men; to find how much 8 Men must pay for 5 Months. The State of the 5 given Terms is, 6 Men — 7 Mo. — 40 *l.* — 8 Men — 5 Mo. The 1st single Question of the *Rule of Three* is, If 6 Men pay 40 *l.* what must 8 Men in the same Time? The Answer is found by multiplying 40 *l.* by 8, and dividing the Product by 6; *that is,* we take the 6th part of 8 times 40, which may be express'd in a general fractional form thus, $\frac{40 \times 8}{6}$ Then the 2d single Question is, If 7 Months cost $\frac{40 \times 8}{6}$ what will 5 Months cost? And the Answer of this is found by multiplying the middle Term $\frac{40 \times 8}{6}$ by 5, and dividing the Product by 7: But a Fraction is multiply'd by multiplying its Numerator, and divided by dividing its Denominator; therefore the Answer is express'd thus, $\frac{40 \times 8 \times 5}{6 \times 7}$; which is the Quote of the continual Product of 40, 8, & 5, divided by the Product of 6 & 7, the Divisors of the two simple Questions: All which is according to the Rule. And, whatever the Question be, 'tis manifest that there will be always the same Reason; for, by expressing the Answer of the 1st simple Question in this general fractional Way, the Answer of the 2d will necessarily be express'd by a Quote made of a Divisor which is the Product of the Divisors of the two simple Questions, and a Dividend which is the Product of the other 3 given Terms.

OBSERVATION *relating to the preceding* RULE.

As I made no Distinction of a *Rule of Three Direct* and *Indirect*, so neither have I in the *Rule of Five*, as is commonly done, to no purpose but to make a needless Difficulty; since *Direct* and *Indirect* can be here understood no otherwise than as they relate to the

the two simple Questions of the *Rule of Three* contain'd in it. But I had this further Reason to make no such Distinction; That all the Questions that come under this Rule may be solv'd by two Applications of the *Rule of Three* that are either both *Direct*, or one *Direct* and the other *Indirect*. Thus, the 5 Terms being stated, and the two Questions of the *Rule of Three* consider'd according to the above Rule, if they are both *Direct*, then it may be other Ways solv'd, so as to make one *Direct*, and another *Indirect*: Or, if one is *Direct*, and the other *Indirect*, by the above Rule, it may be solv'd so, as to make both *Direct*; which is done, in both Cases, by making the 1st, 2d, and 4th Terms (of the 5) the 3 Terms of the 1st Operation; then making the Answer of this with the 3d and 5th (of the 5) the 3 Terms of the 2d Operation.

Exa. 1st. If 7 Men in 3 Months spend 100 *l.* how much will 12 Men spend in 5 Months? Here both the Operations, according to the preceding Rule, are *Direct*; But it may be done thus: (1°) If 7 Men take 3 Months (to spend 100 *l.*) how long will 12 Men take (to do the same)? They must take less Time; therefore this is *Indirect*: Then, whatever time 12 Men take to spend 100 *l.* they will spend more or less in 5 Months, according as 5 is more or less than the Answer of the 1st Question; therefore this is *Direct*.

Exa. 2d. If 7 Men spend 100 *l.* in 3 Months, in what Time will 12 Men spend 48 *l.*? By the preceding Rule the 1st Operation is *Indirect*, and the other *Direct*; but, do it the other Way, and they will be both *Direct*. Thus; if 7 Men spend 100 *l.* (in 3 Mo.) 12 Men will spend more (in the same Time); therefore this is *Direct*. Again; If 12 Men spend the Sum found by the 1st Question in 3 Months, they'll take more or less Time to spend 40 *l.* according as this is more or less than the Answer of the 1st Question; therefore this is also *Direct*.

That the final Answer, or Answer of the 2d Operation, will be the same in both these Methods, will appear from the Nature of the Thing; because both Ways there is a reasonable and natural Connection betwixt the two Operations, which take in all the Circumstances of the Question: Therefore I shall not trouble you with any farther Demonstration of it; and shall only add, That I chuse the first Method, because it leads, in a more easy and plain Way, to the Method of reducing both the Operations to one Division.

OBSERVATION *relating to other* Complex Questions.

All *Complex Questions* that are solvable by two Operations of the *Rule of Three*, so that the Answer of the 1st is a Term of the 2d, tho' they have not all Circumstances like those belonging to the *Rule of Five*, yet if we consider and perceive what are the Terms of these two Operations, 'twill be easy to reduce them to one Division, as we have done those of the *Rule of Five*; for, by expressing the Answer of the 1st Question Fraction-wise (as above) and placing that Expression where it should be in the 2d Question, we shall easily perceive which of the given Numbers are to be multiply'd for a Divisor, and which for a Dividend. I shall illustrate this by an Example: Suppose 14 yards of Cloth at the rate of 8 *s.* for 3 yards, are given for Sugar at the rate of 2 *s.* 8 *d.* (or 32 *d.*) for 5 *lb.* how much Sugar ought to be given? To do this at two Steps, I say, If 3 yards cost 8 *s.* what 14 yards? *Answ.* 37 *s.* 4 *d.* Then, if 32 *d.* buy 5 *lb.* what 37 *s.* 6 *d*? And here, without finding the Answer of the first, I see that 3 and 32 are the Divisors; but then, because the 32 is *d*, make 8 *s.* = 96 *d.* and the 3 Numbers that produce the Dividend are 96, 14, 5.

CHAP.

CHAP. IV.

Rule of Fellowſhip.

DEFINITION.

THIS Rule ſhews how, by two or more Independent Operations of the *Rule of Three*, to divide any given Number into unequal Parts, proportional to certain other given Numbers. 'Tis call'd the *Rule of Fellowſhip*, becauſe the more common and uſeful Application is in the Diviſion of Gains, Loſſes, or other things among Partners in Company: But, as there are alſo other Applications of it, I have made the Definition *Univerſal*; and, for the Solution, this is the

RULE. Add the given Numbers (to which theſe Sought are Proportional) into one Sum, which make the 1ſt Term; make the Number to be divided the 2d or middle Term; and the given Numbers (or parts of the 1ſt Term) make them ſeverally the 3d Terms of ſo many diſtinct Queſtions of the *Rule of Three*; and the Numbers thus found are the Anſwers: The Reaſon of which is manifeſt.

To *prove* the Anſwer to be right, add them all into one Sum; and this ought to be equal to the middle Term, becauſe the Numbers found are the ſeveral Parts of the middle Term, and the Parts muſt be all together equal to the Whole. But *Obſerve*, that if there are Remainders (tho' of the loweſt Species) in this Diviſion by which the Anſwers are found, theſe make Fractions, which are alſo to be added: In order to which, let the Remainders be all of the loweſt Species, then add them, and divide their Sum by the common Diviſor, (which is the common Denominator; and here there muſt be no Remainder) the Quote added to the Sum of the Integral parts of the Anſwer, will make it equal to the middle Term, if the Work is right.

Qu. 1ſt. Two Men (*A*, *B*) make a Common Stock, whereof *A*'s Part is 240 *l.* and *B*'s 360 *l.* After a certain time they make 80 *l.* Gain or Loſs; What is each of their Shares? *Anſw.* *A*'s 32 *l.* and *B*'s 48 *l.*

Stocks	*Operation*		*Gain or Loſs.*
	l. *l.* *l.*	*l.* *l.* *l.*	
A 240 *l.*	If 600 — 80 — 240	600 — 80 — 360	*A.* 32 *l.*
B 360	240	360	*B.* 48
600	600) 19200 (	600) 28800 (	80
	Quote, 32 *l.*	48 *l.*	

Qu. 2d. *A*, *B*, *C* make a Common Stock, whereof *A* has 246 *l.* *B* 392 *l.*: 18 *s.* *C* 278 *l.* They gain or loſe 64 *l.* What is each Partner's Share of it? See the *Anſwers* following.

Stocks

	Stocks
A.	246 l.
B.	592 : 18 s
C.	278 :
	916 : 18

For A.

```
 l.   s.    l.    l.
916 : 18 — 64 — 246
        Or,
  s.      l.      s.
18338 — 64 — 4920
             4920
18338 ) 314880 (17 l.
        18338
        131500
        128366
          3134 rem.
            20
         62680 (3 s.
         55014
          7666. rem.
            12
         91992 (5 d
         91690
           302 rem
             4
          1208 f.
```

For B.

```
  s.     l.    s.        l.   s.
18338 — 64 — 7858 (viz. 392 : 18)
             7858
18338) 502912 (27 l.
       36676
       136152
       128366
         7786 rem.
           20
       155720 (8 s.
       146704
         9016 rem.
           12
       108192 (5 d.
        91690
        16502 rem.
            4
        66008 (3 f.
        55014
        10992 rem.
```

For C.

```
  s.     l.    s.
18338 — 64 — 5560
             5560
18338) 355840 (19 l.
       18338
       172460
       165042
         7418 rem.
           20
       148360 (8 s.
       146704
         1656 rem.
           12
        19872
        18338
         1534 rem.
            4
         6136 f.
```

Shares of Gain or Loss.

	l. : s. : d. : f.		Remainders.
A.	17 : 3 : 5 : 0	—	1208
B.	27 : 8 : 5 : 3	—	10994
C.	19 : 8 : 1 : 1	—	6136
	64 : 0 : 0 : 0	=	18338 (1 f.

Observe: Since the 1st and 2d Terms are the same in all the Operations, if there are many Shares, 'twill be convenient to make a Table for the Divisor or first Term, especially if it's a great Number, and perform the Division as directed in § 2d of *Ch.* 6, *Book* 1*st.* And, for the same Reason, you may make a Table for the middle Term, as a common Multiplicand, and do the Multiplications as in § 2*d* of *Ch.* 5, *Book* 1; which will be convenient, especially if the middle Term consists of more, and a greater variety of significant Figures than the Extremes by which 'tis to be multiply'd. *Remember* also, that this Method can be follow'd only when you are to do the Work of the *Rule of Three* by the General Rule: But, in some particular Cases, you'll find other Contractions more convenient.

Again; where there are many Shares, 'twill be a useful Method to reduce the first two Terms to two others in the same proportion, but whereof the first is 1; which is done *thus:* The given 1st and 2d Terms being Simple Numbers (or made so) find a 4th Proportional to these two and 1; then say, As the 1st to the 2d, so 1 to a 4th; which will be like the middle Term. And in compleating this 4th, instead of the common Method of Reduction, carry it on decimally to more or fewer places, according as the given Parts of the 1st Term are greater or lesser Numbers: Then take this 4th Term for the 2d or

2d or middle Term, and 1 for the 1st, in all the Operations by which you find the Numbers sought. And so they are all found by *Multiplication*, because 1 is the Divisor. And, in these Multiplications, you may usefully apply the Tabular Method explain'd in *Ch.* 5, *B.* 1, observing always that whatever Denomination the 1 has, the 3d Term hath the same.

Thus, in *Exa.* 2*d*, say by the *Rule of Three*, as 916 *l*: 18 *s*. is to 64 *l*. so is 1 *l*. to a 4th Term, which you'll find to be this Decimal of a Pound, *viz*. .0698 &c. And in finding this, I either reduce 916 *l*: 18 *s*. to 18338 *s*. and so the Proportion is as 18338 *s*. to 64 *l*. So is 20 *s*. (equal to 1 *l*.) to .0698 &c. Or, more easily, by expressing the 18 *s*. Decimally it is, as 916.9 *l*. to 64 *l*. so is 1 *l*. to .0698 &c *l*. Then the 1st and 2d Terms, in all the Operations for the Answers are 1 *l*. and .0698 *l*. And then the 3d Terms being all in the Denomination of Pounds, like the 1st Term (by expressing the 18 *l*. decimally) the Answers are found by multiplying the several 3d Terms by .0698, as below, which produce the same Integral Answers as the preceding Method.

Observe: In the 1st and 2d Parts I have made but two Steps in the Multiplication, by multiplying with 24 and 27 at once.

.0698	.0698	392.9
246	278	.0698
4188	5584	31432
16752	18846	35361
		23574
17.1708	19.4044	27.42442
or	or	or
17 *l*: 3 *s*: 5 *d*.	19 *l*: 8 *s*: 1 *d*.	27 *l*: 8 *s*: 5 *d*: 3 *f*.

Again Observe, that we may also find the new middle Term by expressing the mix'd Numbers, not decimally, but by Reduction, saying, As 18348 *s*. to 64 *l*. so is 1 *s*. to a 4th, which will be .00349 &c *l*. which is different from the other: But, in using this, we must also express the 3d Term in Shillings, and then we shall have the same Answers; but the former Method is easiest.

As the Shares of *Gain* or *Loss* are, in these Questions, found by the total Gain or Loss and the particular Stocks; so, after the same manner, we may find the particular Stocks, from the total Stock and the Shares of Gain or Loss.

The following *Questions*, done after the same manner, shew the Application of this Rule to other Subjects.

Qu. 3*d*. *A*, *B*, *C* buy together 638 yards of Cloth, of the Value whereof *A*. paid 20 *l*. *B* 260 *l*. and *C* 480 *l*. How much of the Cloth must each of them have? Add 200 *l*. 260 *l*. 480 *l*. into one Sum 940, and then divide 638 yards in proportion to the given Parts of 940.

Qu. 4*th*. There are 3 Horses *A*, *B*, *C*; in the same Time that *A* can eat 5 Bolls of Oats, *B* can eat 7, and *C* 9. How must 25 Bolls be parted among them, that they may begin and end at the same time? Add 5, 7, and 9, the Sum is 21; then divide 25 in proportion to the given Parts of 21.

Qu. 5*th*. There was a Mixture made of 3 different kinds of Wine, in which for every 3 Gallons of one kind there were 4 of another, and 7 of the third; How much of each kind is in a Mixture of 146 Gallons? Add 3, 4, 7, &c.

Qu. 6*th*. Three Butchers pay among them 40 *l*. for a Grass-Inclosure, into which they put 200 Cows, whereof *A* had 80, *B* 100, and *C* 120; How much ought each to pay? Or, what they pay being given with the total Number of Cattle, we may find how many belongs to each.

Qu. 7*th*. A Father left his Estate of 1000 *l*. among 3 Sons, in such manner that for every 2 *l*. that *A* gets, *B* shall have 3, and *C* 5; How is the Estate to be divided?

Of Fellowship *with* TIME.

When Stocks continue unequal Time in Company, so that a Consideration must be made of the Time, as well as of the Stock, this is call'd *Fellowship with Time*; for which this is the

RULE. Let all the Stocks be of one Denomination, and also the Times; then multiply each Partner's Stock by his Time, and divide the Gain or Loss in proportion to these Products.

Qu. 8th. *A* had in Company 45 *l.* for 3 Months; *B* 58 *l.* for 5 Months; and *C* 92 *l.* for 7 Months; at the end of which they find 48 *l.* gain'd; What is each Partner's Share?

The Products are, for *A* $45\times3=135$; for *B* $58\times5=290$; and for *C* $92\times7=644$; whose Sum is $135+290+644=1069$. Then the Proportions are, as 1069 *l.* to 48 *l.* so is 135, 290, 644 severally to the proportional Shares of 48 *l.*

The *Reason* of this Rule can be no other than an Agreement of Parties, that their Shares of Gain or Loss shall be so proportion'd to one another, as those Sums of Interest which, at any rate *per Cent. per Annum*, might be gain'd by the particular Stocks, in the time of their continuance in the Common Stock. *Now*, that the Rule is agreeable to this Supposition, I thus shew: By multiplying the Particular Stocks and Times, we reduce the Question to another State, *viz.* wherein the Particular Stocks are equal to those Products, and in which therefore the Shares of Gain must be proportion'd to those Products; and the Times all equal to an Unit of the Denomination of Time multiply'd: So 45 *l.* bearing Interest for 3 Months is equivalent to 3 times 45, or 135, for 1 Month, at any Rate of Interest: And so of the rest. Consequently the 48 *l.* gain'd in 7 Months is truly proportion'd to those Products.

Qu. 9th. Suppose *A* put in 40 *l.* and at 4 Months end took out 10 *l.* and at 2 Months thereafter put in 30 *l.* *B* put in 50 *l.* and at 3 Months put in 20 *l.* At 8 Months end they balance their Accounts, and find 18 *l.* gain'd; What is the Share of each?

In such Questions, where each Partner's Stock varies by Addition and Subtraction, we must consider how long each Part of the varying Stocks continued in Company, and multiplying them by their Times, the Sums of these Products are the Numbers by which the Division is to be made; as here.

A had 40 *l.*	then 30 *l.*	then 60 *l.*	}	*B* had 50 *l.*	then 70 *l.*
for 4 *Mo.*	2 *Mo.*	2 *Mo.*	}	for 3 *Mo.*	5 *Mo.*
160	60	120	}	150	350

The Sum of *A*'s several Products is $160+60+120=340$. Of *B*'s is $150+350=$ $=500$. Then $340+50=840$. And as 840 to 18 *l.* so 340, & 500 severally to the Shares of 18 sought.

There are other *Questions* of a kind with these, and wrought the same Way; as, the following.

Qu. 10th. Three Persons, *A*, *B*, *C*, hire together certain Pasture-Ground for 24 *l.* in which *A* keeps 40 Cows for 4 Months: *B* keeps 30 Cows for 2 Months; and *C* keeps 36 Cows for 5 Months: How much of the Rent ought each of them to pay?

Multiply each Person's Number of Cows by the Time they were kept, and by these Products proportion the Rent.

And

And if the Partners take out and put in Cattle at different Times, then work as in *Queſt. 9th.*

To the preceding Queſtions I ſhall add the following Collection, in which the Student will find an Uſeful Exerciſe.

Qu. 11th. *A*, *B*, and *C* make a Stock, whereof *A* has 20*l.* *B* 30*l.* They gain 36*l.* whereof *C* got 16*l.* What was C's Stock, and the Gain of *A* and *B* ?

Take 16 from 36, and the Remainder 20 is the Sum of the Gain of *A* and *B*; which being divided in proportion to their Stocks, gives their Shares: Then find *C*'s Stock in ſuch proportion to his Gain, as *A* or *B*'s Stock to his Gain.

Qu. 12th. *A* put into a Common Stock 20*l.* and B 144 Ducats; they gain'd 60*l.* of which *A* got 38*l.* What was the Ducat valued at ?

Take 38 from 60, the Remainder 22 is *B*'s Gain: Then ſay, As 38 *l.* (*A*'s Gain) to 20*l.* (his Stock) ſo is 22*l.* to a 4th Term, which is *B*'s Stock: Then, if 144 Ducats give that Stock, what's 1 Ducat worth ?

Qu. 13th. *A*, *B*, and C make a Common Stock of 468*l.* with which they trade, and gain a certain Sum, whereof the Shares of *A* and B together make 64*l.* of *B* and C 58*l.* of *A* and *C* 70*l.* What is the particular Stock and Gain of each Partner ?

Add 64, 58, and 70, the Sum 192 *l.* is double the total Gain, becauſe each Partner's Share is twice contain'd in it; therefore the half of it 96*l.* is the total Gain: From which take 64*l.* (*A* and *B*'s Share) the Remainder 32 is *C*'s Share; which taken from 58 *l.* (*B* and *C*'s Shares) leaves 26*l.* for *B*'s Share; which taken from 64*l.* (*A* and *B*'s Share) leaves 38*l.* for *A*'s Share: Then having the particular Gains, divide the total Stock proportionally.

Qu. 14th. *A* has in Stock 35*l.* and *B* 20*l.* They agreed, that the Gain be divided ſo as *A* have 10 *per Cent.* and *B* only 8; How is 40*l.* to be divided betwixt them ?

Find what's due to 35*l.* at the rate of 10 *per Cent.* and to 20*l.* at the rate of 8 *per Cent:* then divide the total Gain 40*l.* in proportion to thoſe Sums; for, the only Meaning ſuch a Queſtion can have is, that the Gain be proportion'd to what 35 would draw of 10 *per Cent.* and 20 of 8 *per Cent.* and not, that *A* has really 10 *per Cent.* and *B* 8, for their Stocks; for they will have more or leſs, according as the total Gain happens to be.

Obſerve. Mr. *Hill*, without expreſſing any particular Stocks, ſuppoſes 120*l.* Gain, and *A* to gain 10 *per Cent.* *B* 8; and, to ſolve the Queſtion, he bids us ſuppoſe their Rates of Gain *per Cent.* to be their Stocks, and in that Proportion to divide 120*l.* but he has neglected to explain ſomething neceſſarily ſuppos'd in this Solution, *viz.* That their real Stocks were equal: In which Caſe, be theſe Stocks what you will, the Gains are proportional to the ſeveral Rates *per Cent.* But, if 10*l.* and 8*l.* are their real Stocks, then the Solution is wrong, and we ought to find what's due to 10*l.* at 10 *per Cent.* and to 8*l.* at 8 *per Cent.* and by theſe Sums proportion the Gain.

Qu. 15th. *A* and B were in Company thus: *A* had 50*l.* in Stock for 10 Months, and *B* had his Stock in for 8 Months, and receiv'd equal Share of the Gain; What was *B*'s Stock ?

Since their Gain was equal, ſo muſt the Products of their Stocks and Times; wherefore multiply *A*'s Stock and Time, *viz.* 50*l.* by 10, the Product is 500; which divide by *B*'s Time 8, the Quote 62*l*: 10 *s.* is *B*'s Stock. Or, which is the ſame, make this Proportion; as *B*'s Time 8 Months to *A*'s Time 10 Months, ſo reciprocally *A*'s Stock 50*l.* to *B*'s 62*l*: 10 *s.*

Obſerve: If we ſuppoſe *A*'s Gain is to *B*'s in any other Proportion, as 2 to 3, then, becauſe the Gains are proportional to the Products of Stock and Time, ſay, As 2 to 3, ſo is 500*l.* (the Product of *A*'s Stock and Time) to a 4th, *viz.* 750*l.* (the Product of *B*'s Stock

and

and Time); which therefore divided by 8 (*B*'s Time) the Quote is 93 *l*: 15 *s*. for *B*'s Stock.

Qu. 16*th.* *A* receives of Gain 20 *l*. for 8 Months, *B* 25 *l*. for 7 Months, and *C* 36 *l*. for 5 Months; the ſum of the Products of their Stocks and Times is 520 *l*. What were their Stocks?

Divide 520 *l*. in 3 parts proportion'd to 20 *l*. 25 *l*. and 36 *l*. then divide each of theſe parts by the reſpective Times, 8 Mo. 7 & 5, the Quotes are the Stocks ſought.

Obſerve: If inſtead of the particular Times the Stocks were given, and the Times requir'd, the Operation is the ſame; for 520 being reſolv'd into 3 parts proportion'd to the Gains, divide theſe parts by the Stocks, and the Quotes are the Times.

Qu. 17*th.* *A* gains 20 *l*. and his Stock is 15 *l*. more than *B*'s, whoſe Gain is 12 *l*. What are the particular Stocks?

Say, As the difference of the Gains is to the difference of the Stocks, ſo is each of the particular Gains to the correſpondent Stocks.

For, ſince the ſum of the Gains is to the ſum of the Stocks as each Gain to its Stock, then, from the nature of Proportion, the difference of Gain is to the difference of Stock as each Gain to its Stock.

Qu. 18*th.* *A* gains 20 *l*. in 6 Months, *B* 18 *l*. in 5 Months, and *C* 28 *l*. in 9 Months, whoſe Stock is 72 *l*. What are the Stocks of *A* and *B*?

Multiply *C*'s Stock and Time, the Product is 648 *l*. Then, as 28 *l*. (*C*'s Gain) to 648 *l*. ſo are 20 *l*. and 18 *l*. to the Products of *A* and *B*'s Stock and Time; which being found, divide them by their Times, and the Quotes are the Stocks.

If, inſtead of the real Sums of Gains, there were given 3 Numbers in the ſame Proportion as the real Gains, the Work is the ſame. Or ſuppoſe, inſtead of the Particular Gains, that *A* has $\frac{1}{3}$ of the whole Gain, and *B* $\frac{1}{5}$, then we muſt add theſe Fractions, and take the Sum from 1, the Remainder is the Fraction of the total Gain which *C* has; and then uſe theſe Fractions as the Particular Gains.

Again; If their particular Gains and Stocks are given, with the Time of one Partner, to find the Times of the reſt, the Work is alſo the ſame.

Qu. 19*th.* *A*, *B*, *C* have a Common Stock of 1000 *l*. *A* gains 100 *l*. for 9 Months, *B* 80 *l*. for 12 Months, and *C* 120 *l*. for 8 Months; What were the Particular Stocks?

Divide each Partner's Gain by his Time, and then divide 1000 *l*. into 3 parts proportion'd to thoſe Quotes. The *Reaſon* of this is, that if the Times are equal, the Stocks are in proportion to the Gains; and if the Gains are equal, the Stocks muſt be reciprocally as the Times; and conſequently neither being Equal, the Stocks are as the Gains directly, and as the Times reciprocally; *that is*, as the Quotes of the Gains divided by the Times. Or, it may be ſhewn this Way: Let g, s, t repreſent the *Gain*, *Stock*, and *Time* of one Partner, and G, S, T thoſe of another; then, becauſe the Gains are in proportion to the Products of Stock and Time (as already demonſtrated) and theſe Products being repreſented by st. ST, it is $g : st :: G : ST$; but by equally dividing the relative Terms, *viz.* g and st by t, and G and ST by T, the Quotes are ſtill proportional; that is, $\frac{g}{t} : s :: \frac{G}{T} : S$.

Obſerve: If inſtead of the total Stock and particular Times (as above) were given the particular Stocks and total Time to find the particular Times, the Solution is after the ſame Way, and for the ſame Reaſon, *viz.* dividing the particular Gains by their Stocks, and proportioning the Times to thoſe Quotes.

Qu. 20*th.* *A* hath 200 *l*. more Stock than *B*, but *A* continued his only 5 Months, and B 9, and drew equal Gains; What are the Stocks?

Say, As the Difference of Times to the Difference of Stocks, ſo is *A*'s Time to *B*'s Stock, and *B*'s Time to *A*'s Stock; or, having one Stock, by that and the Difference find the

the other. The *Reaſon* of this is, that when the Gains are equal, the Stocks are reciprocally as the Times; and therefore, from the Nature of Proportion, the Difference of the Times is to the Difference of the Stocks, reciprocally as the particular Times to the Stocks; *i. e.* as *A*'s Time to *B*'s Stock, or as *B*'s Time to *A*'s Stock.

Qu. 21*ſt.* *A*, *B*, and *C* have 100 *l.* to be divided among them, in ſuch manner that 2 times *A*'s Share be equal to 3 times *B*'s, and 4 times *B*'s be equal to 5 times *C*'s; What are their Shares?

'Tis plain by the Conditions, that as oft as *A* gets 3, *B* muſt have 2; alſo as oft as *B* gets 5, ſo oft muſt *C* get 4: Then I ſay, As 5 to 4, ſo is 2 to $1\frac{3}{5}$, ſo that as oft as *B* gets 2, ſo oft *C* gets $1\frac{3}{5}$; but ſo oft alſo *A* gets 3; therefore the Proportions of the Shares ſought are $3 . 2 . 1\frac{3}{5}$, or 15 . 10 . 8, according to which 100*l.* is to be divided.

Suppoſe the Conditions thus; $\frac{1}{2}$ of *A*'s Share is equal to $\frac{2}{3}$ of *B*'s, and $\frac{2}{3}$ of *B*'s equal to $\frac{5}{7}$ of *C*'s; we may find the Proportions of their Shares the ſame Way as before.

Qu. 22*d.* A Father, ignorant of Arithmetick, orders his Eſtate of 500 *l.* to be divided among three Sons, ſo as the eldeſt get $\frac{1}{2}$, the ſecond $\frac{3}{5}$, and the third $\frac{4}{7}$; What is each Son's Part?

Here 'tis impoſſible to give them theſe Shares, becauſe $\frac{1}{2}$, $\frac{3}{5}$, and $\frac{4}{7}$ exceed the whole; and therefore the Meaning of the Queſtion muſt be underſtood to be, the dividing 500 *l.* into 3 Parts that bear ſuch Proportion to one another as theſe Fractions: And the like is to be underſtood of all Diviſions propos'd in this manner, whether the Parts propos'd exceed, or come ſhort of, the Thing to be divided.

Qu. 23*d.* 'Tis propos'd to divide 300*l.* among 3 Perſons, ſo that *A* get 6 *l.* more than $\frac{1}{2}$, *B* 12*l.* more than $\frac{1}{3}$, *C* 8 *l.* leſs than $\frac{2}{3}$; What gets each?

According to the moſt obvious ſenſe of this Queſtion, the Meaning of it is, that the Shares be in proportion to the Sum of 6 *l.* and $\frac{1}{2}$ of 300 *l.* for *A*; 12 *l.* and $\frac{1}{3}$ of 300 *l.* for *B*; and $\frac{2}{3}$ of 300*l.* wanting 8 *l.* for *C*: But *Jeake* (from whom I take it) underſtands it in another Senſe, which indeed I think no Body could ever find in it, as 'tis propos'd; *viz.* that the Shares be ſuch, as if 6 *l.* be taken from *A*'s, 12 *l.* from *B*'s, and 8 *l.* added to *C*'s, the Remainders in the former, and the Sums in this, be to one another as $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$, and ſo the Solution is made thus: Take 6 and 12 from 300, and to the Remainder add 8, then divide this Sum in 3 Parts proportional to $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{2}{3}$, and to theſe Shares add and ſubtract the Sums propos'd. The *Reaſon* of the Work is plain, according to the Senſe he puts upon it.

Qu. 24*th.* Three Perſons, *A*, *B*, and *C*, buy a Ship, of the Price whereof *A* paid $\frac{5}{8}$, *B* $\frac{3}{7}$, and *C* 140 *l.* How much Money paid *A* and *B*? and, What Part of the Ship had *C*?

Add the Fractions $\frac{5}{8}$ and $\frac{3}{7}$, and take the Sum from 1, the remainder is the Part of the Ship belonging to *C*; then ſay, If *C*'s Part coſt 140 *l.* what coſt the Sum of *A* and *B*'s Parts? And having found that, divide it into 2 Parts, proportion'd to one another as $\frac{5}{8}$ to $\frac{3}{7}$.

Qu. 25*th.* There were at a Feaſt 20 Men, 30 Women, and 15 Servants; for every 10 *s.* that a Man paid, a Woman paid 6, and a Servant 2; How much did every Man, Woman, and Servant pay of 24 *l*?

Multiply 20 by 10, 30 by 6, and 15 by 2; then divide 24 *l.* in 3 parts proportion'd to theſe Products (*viz.* 200, 180, and 30) and you have the Total paid by the 20 Men, 30 Women, and 15 Servants: Each of which Sums being divided by their reſpective numbers of Perſons, gives the Payment made by each Individual.

Suppoſe the Conditions ſuch, that a Man pays 3 times as much as a Woman, and 2 *s.* more; that a Woman pays double of a Servant, and 1 *s.* more; To find their Shares, multiply 2 by 20, and 1 by 30, the Products 40 and 30 equal to 70 *s.* take from 24 *l*, the

Remainder is 20*l*: 10*s*. Then, because a Man pays triple of a Woman, suppose a Man pays 3, a Woman pays 1; and because a Woman pays double of a Servant, if a Woman pays 1, a Servant pays $\frac{1}{2}$; so their Proportions are 3 : 1 : $\frac{1}{2}$, or, in whole Numbers, 6 . 2 . 1; *that is*, 6 for a Man, 2 for a Woman, and 1 for a Servant. Multiply these by their respective Numbers of Persons, the Products are 120 for 20 Men, 60 for 30 Women, and 15 for 15 Servants: Then divide 20*l*: 10*s*. in 3 parts, in proportion to one another as are 120, 60, and 15, and divide these parts by their respective numbers of Men, Women, and Servants, the Quotes are what each Man, Woman, and Servant pays of the 20*l*: 10*s*. Lastly, to a Man's Share of this add 2*s*. to a Woman's 1, and you have their compleat Payments of the whole 24*l*.

Observe: If, instead of adding, it had been propos'd to subtract, as if a Woman pays 1*s*. less than the double of a Servant, then add 30*s*. to 24*l*. (subtracting what a Man pays more than triple of a Woman); and, in the last Part, instead of adding, subtract 1 from the Woman's Part of the Sum divided.

Qu. 26*th*. A Father dying, left his Wife with Child, to whom he bequeath'd, if she had a Son, $\frac{1}{3}$ of his Estate, and $\frac{2}{3}$ to the Son: But, if she had a Daughter, $\frac{1}{3}$ to her, and $\frac{2}{3}$ to her Mother. It happen'd that she had both a Son and a Daughter; How shall the Estate be divided to answer the Father's Intention?

As the Father plainly design'd the Son to have double of the Mother's Part, and the Mother double of the Daughter's Part, therefore for every 1 the Daughter got the Mother must have 2, and the Son 4; and in proportion to these Numbers 1 . 2 . 4 must the Estate be divided.

[This is a Question propos'd by a *Roman* Lawyer, in the 28th Book of the *Digests*, which he thinks is justly solv'd after this manner.] Again,

Suppose that the Mother had a Son and Daughter who liv'd, but her self dy'd in the Birth; How is the Estate to be divided betwixt the Son and Daughter? *Nicholas Tartaglia* makes this Supposition in his *Arithmetick*, and solves the Question thus; says he, "Had the Mother liv'd, the Proportions are 1 . 2 . 4, as above, therefore the Estate "must be divided in proportion of 1 for the Daughter to 4 for the Son. But I doubt the Justice of this Solution; for tho' this Proportion betwixt the Son and Daughter's Parts, in case of the Mother's Life, is a Consequence of the Father's plain Intentions, with respect to the Mother and Son or Daughter, yet never having a Son and Daughter both together in his view, this Solution seems to have no Foundation. And I rather think the Solution ought to be thus: Find the Parts belonging to Mother, Son, and Daughter, then divide the Mother's Part betwixt the Children, according to the Rule of Heirship in the Country where the Question arises.

Under this Head of *Fellowship* are also comprehended the Calculation of Gains or Stocks betwixt a Merchant and Factor.

Questions *of* FACTORSHIP.

Question 1. A Merchant delivers to his Factor 100*l*. allowing him to join to it 30*l*. and values his Service worth 40*l*. what share of the gain ought the Factor to have?

There are two ways of solving this Question: The generality of Authors do it thus, Add 30*l*. to 40*l*. the Sum is 70*l*. then divide the Gain in two Parts, in proportion as 100*l*. to 70*l*.

Another Method is this; Subtract 40 from 100 which leaves 60, and proportion the Shares of Gain to 60 (for the Merchant) and 70 (*i. e.* 40 and 30) for the Factor.

If the Merchant and Factor determine the meaning of their Agreement to either of these ways of stating the Proportion, there is no more Question; but without this the last Method seems the more reasonable, because the Gain is made upon the real Stock 130 *l.* and not upon the imaginary one 170 *l.* And the more obvious sense of valuing the Factors Service at 40 *l.* seems to be, the allowing him the Gain of 40 *l.* of the real Stock more than what he actually puts in, which must consequently be deducted from the Merchants Stock, and added to his.

Quest. 2. A Merchant's real Stock being 100 *l.* and the Factors 30 *l.* who received $\frac{1}{3}$ of the Gain: What was his Service valued at?

To proceed upon an imaginary Stock, say as $\frac{2}{3}$ to $\frac{1}{3}$, or as 2 to 1, so is 100 to 50, from which take 30, the remainder 20 is the Answer.

But upon the real Stock, find the 3d part of 130, from which take 30, the remainder is the Answer.

Quest. 3. A Merchant's real Stock being 100 *l.* and the Factors Service valued at 20 *l.* who received $\frac{1}{2}$ of the Gain; What was the Factor's real Stock?

To proceed upon an imaginary Stock, it is 80 *l.* because 20 and 80 makes 100 equal to the Merchant's. By the other Method it is only 60, because 20 and 60 make 80, the half of 160, the total real Stock.

Quest. 4. The Merchant's real Stock being 100 *l.* and the Factor being allowed $\frac{1}{4}$ of the Gain for his Service, what real Stock must he join to have $\frac{1}{3}$ of the Gain?

When the Factor gets $\frac{1}{4}$ (without any real Stock) his Service is there valued at 25 *l.* the 4th of the real Stock 100 *l.* or 33 *l.* 6 *s.* 8 *d.* the $\frac{1}{4}$ of the imaginary Stock 133 *l.* 6 *s.* 8 *d.* found by adding the $\frac{1}{3}$ of 100 *l.* to 100 *l.* Then with this Value of his Service, proceed to find the real Stock that he must have to get $\frac{1}{3}$ Gains by the Methods of Question 3d.

Observe, that in all the preceding Questions we may suppose 2 or more Merchants with the Factor; it will be easy to apply the same Rules, by adding the Stocks of all the Merchants into one Sum, and considering that as one Stock; and then, besides whats already demanded, it may also be demanded to find the Gain of each Merchant; thus, by the first Method of an imaginary Stock, what remains to the Merchants after the Factor's part is deducted, must be divided in Proportion to their real Stocks.

By the real Stocks we must divide the Factor's estimation into parts proportioned to the Merchants real Stocks, and take the parts answering to each from itself, the remainders are the Numbers by which the Merchants shares are to be proportioned.

Quest. 5. A Merchants real Stock being 120 *l.* and the Factors 60, they agreed, that at a Years end the Factor should have $\frac{1}{2}$ of both the Stocks and Gain, but they broke up at 8 Months end, having gained 150 *l.* How much ought the Factor to have?

Here 'tis plain the Factor for 12 Months Service was to have not only the Gain of 30 *l.* of the Merchants Stock, but also 30 of the Stock itself; so that his Service was valued at 30 *l* real Stock; but the Society lasting only 8 Months, 'tis plain he ought only to have 20 *l.* Pound (which is to 30 *l.* as 8 Months to 12) and this added to his own 60, makes 80 *l.* which he receives of the real Stock; and the Merchants part being 100 *l.* then it is as plain the 150 *l.* Gain must be divided into 2 parts, proportion'd to these Stocks 80 and 100.

Observe, *Buteo*, from whom I take this Question, finds fault with this Method of solving it (which he says both *Lucas* and *Stephanus* have followed) and does it himself thus.

He says, that since upon supposition of the Society continuing 12 Months, the Factor was to have the half, therefore his Service was valued at 60, to make his real Stock of 60 equal to the Merchants 120; so that the Society continuing only 8 Months, his estimation is only 40, which added to his real Stock 60, makes 100; and the Merchants being 120: Therefore the Sum of the real Stocks (*viz.* 120 and 60) and the Gain (150, which makes 330) ought to be divided into 2 Parts, proportional to 100 for the Factor, and 120 for the Merchant.

Now

Now all the fault that *Buteo* finds with the other Method, is, that he sees no Reason why the Merchants Stock of 120 should be diminished; but to me the Reason is obvious, because, tho' he puts in 120, yet part of it belongs to the Factor for his Service; and if the Society had continued 12 Months according to the Conditions, 30 *l.* of the Merchants 120 would have been given to the Factor; and for the same Reason the Society continuing 8 Months, 20 of the Merchants must be given to the Factor, and Gain proportionally.

But, in his way of solving the Question, he estimates the Factor's service otherwise than the plain Conditions of the Question; and so brings in an imaginary Stock, and by Consequence a false Proportion.

CHAP. V.

QUESTIONS *concerning* Loss *and* Gain.

QUEST. 1. A Parcel of Goods being bought for 60 *l.* and sold for 75 *l.* what was the rate of Gain *per Cent?* I say if 60 *l.* gain 15 *l.* What will 100 *l.* Gain. the answer is 5 *l.*

Quest. 2. Having bought 18 Gallons of Brandy for 12 *l.* how may I sell 1 Gallon, and Gain at the rate of 8 *per Cent?* I find what 1 Gallon cost, it is 13 *s.* 4 *d.* Then I say if 100 give 108, what will 13 *s.* 4 *d.* give? it is 14 *s.* 4 *d.* 3 *f.* $\frac{21}{25}$.

Quest. 3. Having sold 11 Yards of Cloth for 4 *l.* 16 *s.* and thereby Gain'd at the rate of 10 *per Cent.* What was the prime Cost of 1 Yard? First I find 1 Yard is sold for 9 *s.* 6 *d.* then if 110 comes of 100 (prime cost) To what prime cost, at that rate, does 9 *s.* 6 *d.* belong? *Answer.* 8 *s.* 7 *d.* 2 $\frac{6}{11}$ *f.*

Q. 4. Having sold 2 Yards of Cloth for 11 *s.* 6 *d.* I gained at the rate of 15 *per Cent*; but had I sold it for 12 *s.* what is the rate of Gain *per Cent?* I say as 11 *s.* 6 *d.* is in proportion to 115 *l.* so is 12 *s.* to a 4th Term, which I find to be 120 *l.* and so 20 *l.* is the Answer of the Question.

Observe. This Question is in Substance, and Numbers, the same with Mr. *Hill*'s 8th Question of Loss and Gain; but neither his Operation nor Answer are the same. He states it thus; As 11 *s,* 6 *d.* is to 15 *l.* so is 12 *s.* to 15 *l.* 13 *s.* with a very small Fraction. But this state of the Proportion is quite wrong; because 11 *s.* 6 *d.* and 15 *l.* are not similar Terms; the 1st being a Sum of prime Cost and Gain put together, and the other only an Article of Gain; which shews that the 2d Term ought to be 115 *l.* which is the sum of the prime Cost 100 *l.* and is the Gain made upon it at the same rate with the Gain included in 11 *s.* 6 *d.* Whoever understands the nature of Proportion, will find no difficulty to perceive the Reason of this; yet for the sake of others, I'll shew, by another way of Solving the Question, that the first is the true Answer. Thus, from the 1st Sale and rate of Gain find the prime Cost of 1 Yard, it is 10 *s.* (for as 15 to 100, so is 11 *s.* 6 *d.* to 10 *s.*) then the 2d Sale being 12 *s.* the Gain is 2 *s.* therefore say, If 10 Gain 2, what will 100 Gain? It is 20.

In the following Questions I consider the Forbearance of *Money*; or *Time* allowed for Payment.

Quest 5.

Qu. 5. Having bought a parcel of Goods for 18*l.* and sold the same immediately for 25 *l.* with 4 Months Credit, What is gain'd *per Cent. per Annum?* Say by the Rule of *Five*, If 18*l.* in 4 Months gain 7*l.* what will 100*l.* gain in 1 Year?

Observe 1°. If the Gain *per Cent. per Ann.* is given (suppose 12 *per Cent.*) to find the Time that ought to be allow'd, then say, If 100*l.* gain 12 *l.* in 1 Year, in what Time must 18*l.* gain 7*l*?

(2°) Or, if the Rate of Gain is given, with the prime Cost and Time, to find the selling Price, say, If 100*l.* in 1 Year gain 12*l.* what must 18*l.* gain in 4 Months? Which is to be added to the prime Cost.

(3°) If the Rate of Gain, Time, and selling Price are given to find the prime Cost, as, suppose 4 Months allow'd for payment of 25*l.* by what was gain'd at the rate of 12*l.* *per Cent. per Annum*, to solve this, you must first find 4 Months Interest of 100*l.* at the propos'd Rate, which add to 100, then say, As that Sum is to 100, so is 25*l.* to a 4th Term; which is the Sum sought.

Qu. 6. Having bought 40 Gallons of Brandy at 3 *s. per* Gallon, by an Accident there was lost of it 6 Gallons, at what rate *per* Gallon may I sell the rest, with 8 Months Credit, and gain upon the whole prime Cost at the rate of 10 *per Cent. per Annum?* Find the Value of 40 Gallons at 3 *s.* 'tis 6*l*; then say, If 100*l.* in 1 Year gain 10*l.* what will 6 *l.* gain in 8 Months? Add that Gain to 6 *l*, the Sum is the Value at which the whole remaining 34 Gallons are to be sold with 8 Months Credit: From which find the Price of 1 Gallon.

Qu. 7. Having paid 14 *s.* for each of 100 yards of Cloth, I propose to gain 25 *per Cent.* ready Money; and if I sell it upon Time, to have moreover 10 *per Cent. per Annum* for the Forbearance: What must be the Price of 1 yard with 6 Months Credit, to make both those Gains? First find the ready Money Price of the whole at 25 *per Cent.* Gain; then find what Gain or Interest that will give in 6 Months at the rate of 10 *per Cent. per Ann.* which being added to it, the Sum is the Price of the whole forborn 6 Months; by which find the Price of 1 yard.

CHAP. VI.

Questions of Bartering.

Qu. 1*st.* A Gives 4 Hogsheads of Wine, at 9*l. per* Hogshead, to *B*, for Raisins at 7 *d* *per* Pound; How much weight of Raisins ought *A* to receive?

Say, If 1 Hogshead give 9*l.* what will 4? The Answer is, 36*l.* Then, if 7*d.* buy 1 *lb.* what will 36*l.* buy? *Answ.* $102\frac{6}{7}$ *lb.*

Observe this Fraction $\frac{6}{7}$, if it cannot be given in real Quantity by reducing to lower species, then *A* must give 102*lb.* Raisins and 6 *d.* in Money, because $\frac{6}{7}$ of 1 *lb* weight is 6*d.* And, in all Cases, when any Fraction of a Quantity cannot be given, its Value in Money must be given.

Qu. 2*d.* If I get 120 Gallons of Brandy, at 4 *s.* the Gallon, for 28 Bolls of Wheat, how is 1 Boll sold?

Say, If 1 Gallon cost 4*s.* what 128 Gallons? The *Answer* is, 480 *s.* Then, if 28 Bolls cost 480 *s.* what 1 Boll?

Qu. 3*d.* *A* and *B* barter thus: *A* gives 120 yards of Cloth, such that 3 yards $\frac{1}{2}$ cost 15 *s* : 9 *d.* *B* gives part Stockings at 7 *s.* the pair, part Hats at 6 *s.* 6 *d.* a piece, and gives an equal number of Hats and pairs of Stockings; How many were of each?

Find

Find the total Value of *A*'s 120 yards of Cloth, then add the Value of 1 Hat and 1 pair of Stockings; and by the Sum divide the Value of the Cloth, the Quote is the Answer: And if there is a Remainder, it shews that the Value of the 120 yards cannot be exactly given, according to the propos'd Conditions; and therefore, besides the Number expres'd by the Quote, *B* must give *A* so much Money as that Remainder express'd in the denomination of the Dividend; *i. e.* so many *l.* or *s.* or *d.* as the Dividend is, because that is the Value of such a Fraction of one Hat and one Pair taken together.

Observe: If the proportion of the Number of Hats and Pairs of Stockings is suppos'd to be any other than that of Equality; *Exa.* as, 2 Hats for 5 pair of Stockings, then we must add the Value of 2 Hats to the Value of 5 pair of Stackings, and make that Sum the Divisor: Then the Quote being multiply'd by 2, gives the number of Hats, and by 5, gives the number of pairs of Stockings; the Remainder is to be taken the same way as before.

Qu. 4th. *A* and *B* barter thus: *A* has 27 yards Silk-stuff worth 2 *s.* ready Money, but in Barter he will have 2 *s.* 3 *d.* *B* has Hats worth 7 *s.* a piece ready Money; How many of them must he give to *A* for his 27 yards of Stuff? and, What is the Price of a Hat in Barter to equal the advancement of *A*'s Price?

Find the Number of Hats by the ready Money Prices, and for the rais'd Price of the Hats say, As 2 *s.* to 2 *s* : 3 *d.* so is 7 *s.* to the Answer.

Observe: When the Prices of each Party are rais'd proportionally, then it's manifestly indifferent to find the Quantity sought, either by the ready Money Prices or the advanc'd Prices; and therefore the finding B's advanc'd Price in order to find the Quantity, is no ways necessary: as most of our Authors seem to think, by their taking this Method; which has led them into Mistakes, as you'll find below.

Qu. 5th. *A* hath 100 yards Camblet at 16 *d. per* yard ready Money, which he puts away in Barter at 18 *d.* to *B*, taking of him Stockings at 5 *s.* the pair, which are worth but 4 *s* : 6 *d.* ready Money: How many pairs must he give? and, Which of them gains, also how much, by the Bargain?

If the Ready-money and Barter Prices were proportional, neither Party could gain; but the asking *who gains* supposes it's otherwise, or at least uncertain: And therefore we must find the number of Stockings by the advanc'd Prices at which the Barter was actually made. Then find the total Value of the 100 yards at 16 *d.* and of the number of Pairs (found) at 4 *s* : 6 *d*, the Comparison of these Values shews who gains, and how much. And *observe* also, that if there is a Remainder in finding the number of Pairs, as that is so many Pence to be given by *B* to *A*, so, in comparing the real Values, it must be added to the Value of *B*'s Stockings. But if there is no Remainder, you may find the Gains thus; *A* gets 2 *d.* a yard advance, which is 200 *d.* upon the whole: *B* gets 6 *d.* a pair advance, which multiply by his number of Pairs, and compare that Product with 200 *d.* Or you may take another Method, *thus*; say, If 16 *d.* is advanc'd to 18 *d.* what ought 4 *s.* 6 *d.* to be advanc'd to? *Answ.* 5 *s* : 3 *f.* whereby it's plain B is the Loser, who puts his Stockings away at 5 *s.* whereas he ought to have 5 *s* : 3 *f.* so he loses 3 *f.* on every Pair; which multiply'd by the number of Pairs, the Product is what he loses on the whole.

Qu. 6th. Two Merchants have various kinds of Goods to barter; *A* has 735 yards *Indian Silk* at 8 *s* : 6 *d per* yard ready Money, and in Barter 10 *s.* also 532 Canes at 3 *s.* a piece ready Money, and in Barter 3 *s* : 4 *d.* also 16 Pieces of Muslin at 4 *l.* the Piece ready Money, and in Barter 4 *l* : 10 *s.* *B* has scarlet Cloth at 1 *l. per* yard ready Money, Glass Manufacture at 1 *s* : 8 *d. per* Pound-weight ready Money, and a finer kind at 2 *s* : 4 *d.*

How

How many yards of Cloth, and pounds of each kind of Glass, of each a like number, (*i. e.* of yards, and pounds of each Glass) must *B* give to *A*, advancing his Goods proportionally also in Barter?

This Question I take from Mr. *Hatton*, who solves it in this manner, *viz.* He finds *A*'s whole Goods to be worth, at the Barter Price, 528 *l* : 3 *s* : 4 *d.* then he takes the sum of the Prices of 1 of each of *A*'s Things (*viz.* 1 yard Silk, 1 Cane, and 1 Piece of Muslin) at the Ready-Money Price, which make 4 *l* : 11 *s* : 6 *d*; and also at the Barter Price, which make 5 *l* : 3 *s* : 4 *d.* the Difference of those Sums is 11 *s* : 10 *d.* He takes also the Value of an Unit of each of *B*'s Things, which make 1 *l* : 4 *s.* Then he says, If 4 *l* : 11 *s* : 6 *d.* advance 11 *s* : 10 *d*, what ought 1 *l* : 4 *s.* to advance? The Answer is, 3 *s* : 1 *d.* and $\frac{270}{1898}$, which makes the Sum of the Barter Price of a Unit of each of the Things to be 1 *l* : 7 *s* : 1 $\frac{270}{1898}$; And, lastly, by this dividing *A*'s total Value at the Barter Price, the Quote is $389\frac{36280}{35712}$; and so many Units of each of his Things ought *B* to give.

But this Solution is false; which probably I had taken little Notice of, had not the Author made it very remarkable, by complaining how little such Questions are understood; and, that as he seems to think this a Curious and Useful one, so he tells us plainly he expects the Thanks and Applause of the Publick for it; and, that we may see he deserves it, makes what he calls a *Demonstration* of the Truth of his Solution: But I shall easily shew the Error of it: in order to which, I shall first propose

Another Solution. Find the total Value of *A*'s Things at the Ready-money Price, it is $9123\frac{1}{2}$. *s.* which divide by 24 *s.* the Sum of 1 of each of *B*'s Things at the Ready-money Price, the Quote is 380.1458 &c, and so many of each of his Things ought *B* to give to *A*, which is considerably less than the other Answer.

Now, since it be an undoubted Truth, that if the Price of each Party's Goods are advanc'd proportionally, the same quantity must come out, whether we calculate by the Ready-money or the Barter Prices; and the Ready-money Prices being given, there can be no Error in this Method: Therefore the last Answer must be the true one.

This is sufficient to shew, that Mr. *Hatton*'s Answer is wrong: And it shews us also, that the Error must lie in his Way of calculating the Advancement upon *B*'s Things (for, if that were right, his Solution and mine would bring out the same Quantity). But I shall more particularly explain the Error of his Way of finding *B*'s Advancement, and then shew the True Way.

In the first place, it can't be deny'd that since the Barter ought to be equal, therefore the total Value of all that *A* and *B* deliver ought to be equal, both in the Ready-money and the Barter Prices, and consequently the Advance on the wholes (or what the Value of the Barter Prices exceeds that of the Ready-money Prices) ought also to be equal.

Again; If the number of *A*'s Things were equal, then every 4 *l* : 11 *s* : 6 *d.* (the Value of one of each) which is contain'd in the total Value, would advance 11 *s* : 10 *d.* and then of consequence every 1 *l* : 4 *s.* (the Value of one of each of *B*'s Things; that are equal in Number) would advance in the same proportion: But *A*'s Things not being equal in Number, every 4 *l* : 11 *s* : 6 *d.* does not advance 11 *s* : 10 *d*; for, by taking 1 of each, we can only make 16 setts in each of which there are 3 Things whose total Value is 4 *l* : 11 *s* : 6 *d.* because there are but 16 Pieces of Muslin; then there remains 516 Canes, and 719 yards Silk, of which we can make 516 setts of 2 Things, each of which setts advance 1 *s* : 10 *d.* (the Difference of the Ready-money and Barter Prices of 1 yard Silk and 1 Cane); and, lastly, there remain 203 yards Silk, each of which advances 1 *s* : 6 *d.* Therefore the Advance of 1 of each of *B*'s Things can't be truly found by any of these Advances.

But

But this will be further clear by another Unquestionable Method of finding the Advance on a sett of 1 of each of B's Things, which is this; Since the total Value of *A* and *B*'s Things must be equal, both at the Ready-money and Barter Prices, and consequently the total Advance equal, therefore I find the total Value of *A*'s Things at the Ready-money Price, which is $9123\frac{1}{2}$ *s.* and also at the Barter Price, which is $10563\frac{1}{3}$ *s.* the Difference is $1439\frac{5}{6}$ *s.* the total Advance. Again; Since *B*'s Things are equal in Number, therefore 'tis plain that the Value of every sett of 1 of each makes the same Advance, and which must be a proportional part of the total Advance to the total Value, because 1 of each is an Aliquot of the whole; and therefore, I say, if the total Value at the Ready-money Price $9123\frac{1}{2}$, Advance $1439\frac{5}{6}$, what will the Value of 1 of each of *B*'s Things (*viz* 24 *s.*) advance? And I find 3 *s* : 9 *d.* $\frac{24687}{54741}$, which makes the advanc'd Value of a sett of 1 of each of *B*'s Things to be 27 *s* : 9 *d* $\frac{24687}{54741}$ instead of 27 *s* : $1\frac{270}{1098}$ *d.* as Mr *Hatton* makes it. And, to conclude, if we seek *B*'s Quantity by the Barter Price now found for 1 of each, 'tis the same as already found by the Ready-money Prices; for the total Value at the Barter Price (*viz.* $10563\frac{1}{3}$) being divided by 27 *s* : 9 *d* $\frac{24687}{54741}$, the Quote is 383.1458 &c. as before. And thus the two Methods, by the Ready-money and Barter Prices, confirm one another, when the Barter Price is truly found; and finish the Demonstration of Mr. *Hatton*'s Error.

Observe again, That if it be demanded what is the Advance upon a single one of each of *B*'s Things, this has no determinate *Answer*; because it may be any Thing we please, so that the Sum of the Advances upon a single one of each kind be equal to the Advance found for one of each taken together.

Observe also, that if instead of an equal number of each of *B*'s Things we suppose their Numbers to be in any other proportion, as 2 . 3 . and 4; then take the Value of 2 of the one kind, 3 of the other, and 4 of the other; add all these into one Sum, and find the Advance upon that Sum by the Advance of the whole; *thus* you get the Barter Price for 2 of one kind, 3 of another, and 4 of another, taken all together: And by this dividing the total of *A*'s Goods, the Quote shews how many times we are to take 2 of one, 3 of another, and 4 of another, and consequently how many of each.

Qu. 7th. *A* and *B* barter thus; *A* hath 100 yards Cloth at 12 *s.* a yard Ready-money, but in Barter he will have 13 *s* : 6 *d*, and will have also $\frac{1}{4}$ of the Barter-value in Ready-money. *B* hath Sugar at 8 *d.* a pound; How much Sugar ought *B* to deliver? and, How is it to be rais'd to equal the Barter?

Find the total Value of 100 yards Cloth at 13 *s* : 6 *d.* then take $\frac{1}{4}$ of it; which being the Money *B* is to pay, find how much Cloth at 12 *s.* that will buy, and subtract it from 100 yards. Then find how much Sugar at 8 *d.* ought to be given for that Remainder of Cloth at 12 *s.* For raising the Price of the Sugar, 'tis plainly in proportion as the Cloth is rais'd, *i. e.* As 12 *s.* to 13 *s* : 6 *d.* so is 8 *d.* to the advanc'd Price of the Sugar.

Observe: If in this Question the quantity of *A*'s Goods is not given, there can nothing be requir'd but to find how *B* is to advance his Price to equal the Barter.

Such a Question I find in several Authors, but they all solve it after another manner, which is this; They take the propos'd part as $\frac{1}{4}$ (which is to be paid in Money) of *A*'s Barter Price, and subtract it both from that Price and the Ready-money Price; then say, As the Remainder of the Ready-money Price to that of the Barter Price, so is *B*'s Ready-money Price to his Barter Price sought. But this Solution is without Foundation; for there appears no Reason for advancing *B*'s Price in any other proportion than *A*'s; as by this Method it is rais'd in a greater proportion. But if this Method is reasonable at all, it will be so also when the quantity of *A*'s Goods is given to find what *B* must deliver. In which Questions these Authors always use the Barter Price to find the Quantity sought. Let us then apply this *Supposition*, with their Method, to the preceding Question. It's plain, in the first place, that as much Ready-money as *B* pays to *A*, he ought

to

to have Cloth for it at the Ready-money Price, because so much is not barter'd, but bought for Ready-money: And then, for the remainder of the Cloth we are to find how much Sugar must be given for it; and if we do this by a Barter Price greater in proportion to the Ready-money Price than *A*'s is, there's a manifest Injustice done to *A*. I shall *observe* in the last place, that if for the Ready-money which *B* pays he gets Cloth from *A* at the Barter Price, then indeed the Barter Price of *B*'s Goods must be found, according to this other Method, in a greater proportion than *A*'s; which will make the Barter equal, by correcting the Injustice done to *B*, in giving him Goods at a greater than the Ready-money Price, even when he pays Money for 'em: But this Method is a ridiculous going round about to no purpose, and committing two Errors, or doing two pieces of Injustice, that one may correct the other, when there is a more simple and natural Way of doing.

Qu. 8th. *A* has 40 pair of Stockings at 3 *s.* Ready-money, or 3 *s* : 8 *d.* in Barter; but he is willing to discompt 3 *per Cent.* of his Barter Price, to have ¼ of it paid in Ready-money. *B* has Cloth at 10 *s. per* yard Ready-money; How many yards must he deliver, with the Money that *A* requires? and, What is the Rate of his Cloth to equal the Barter?

Take 3 from 100, and say, As 100 to 97, so is 3 *s* : 8 *d.* to a 4th Number which is to be taken for the Barter Price; then do the rest of the Work as in the preceding Question. But if *A* will, besides his Ready-money, gain 3 *per Cent.* say, As 100 is 103, so is 3 *s* : 8. to a 4th Number which is to be taken for his Barter Price.

Qu. 9th. *A* barters with *B* 40 *lb.* of Cloves at 6 *s.* the pound Ready-money, and 7 *s* : 6 *d.* in Barter, but is willing to lose 10 *per Cent.* to have ⅓ Ready-money: What is the Ready-money Price of 1 yard of Velvet deliver'd by *B* at 21 *s.* to equal the Barter? and, How much was deliver'd?

Say, As 100 to 90, so is 7 *s* : 6 *d.* to 6 *s.* 9 *d.* which is the true Barter Price after the Discompt of 10 *per Cent.* Then find how much Velvet (at 21 *s.*) is equal in Value to 40 *lb.* of Cloves at 6 *s.* 9 *d.* And for the Ready-money Price of the Velvet say, As 6 *s* : 9 *d.* to 6 *s.* so is 21 *s.* to the Thing sought.

I take this Question from Mr. *Hill*, with this difference; that I suppose *A*'s Quantity to be given, and *B*'s sought, which he does not; his Demand being only to know the Ready-Money Price of the Velvet; which he finds thus: Having found the 6 *s* : 9 *d.* as before, he takes ⅓ of 7 *s* : 6 *d.* (*viz.* 2 *s* : 6 *d.*) from it self, and also from 6 *s* : 9 *d.* the Remainders are 5 *s.* and 4 *s* : 3 *d.* then says, As 5 *s.* to 4 *s* : 3 *d.* so is 21 *s.* to the Thing sought. This Method is in general like that censur'd above, in Question 7*th*; but it's yet farther wrong, and can be brought to no sense; for the ⅓ Ready-money which *A* demands can be understood no other way than as ⅓ of that Price at which he is willing actually to put away his Cloves, which is 6 *s* : 9 *d.* (*viz.* 7 *s* : 6 *d.* deducing 10 *per Cent.*) and therefore we are to take ⅓ of 6 *s* : 9 *d.* and not of 7 *s* : 6 *d.* Again; by taking that ⅓, (whether it be of 7 *s* : 6 *d.* or 6 *s* : 9 *d.*) from each of them, he does not bring in *A*'s Ready-money Price at all into the Calculation, and so it may be suppos'd to be any thing we please; whereby the same *Answer* will be found in all Suppositions, which is absurd. The Method upon his General Principle ought to have been this, *viz.* Take ⅓ of 6 *s* : 9 *d.* from it self, and from 6 *s.* then, as 4 *s* : 6 *d.* to 3 *s* : 9 *d.* so 21 *s.* to the Thing sought.

I was the more surpriz'd at his Method of solving this Question, that in another Question, wherein *A* proposes to gain 10 *per Cent.* and have ⅓ Ready-money, he proceeds in the Way last directed, *thus*; Having found the Price at which *A*'s Goods are put away (with the 10 *per Cent.* included) he takes ⅓ of that from it self and the Ready-money Price, and by these Remainders finds *B*'s Barter Price, which is the Thing sought in that Question.

Qu. 10*th*. *A* has 100 yards Cloth at 8 *s*. Ready-money, and in Barter 10 *s*. *B* has Raisins at 6 *d*. *per* pound Ready-money, and he will have ¼ of what he puts away paid in Money; How much Raisins must he deliver? and, What Rate do they bear in the Barter? also, How much Money must *A* give *B*?

1°. 100 yards at 8 *s*. is worth 40 *l*. Then find how much Raisins at 6 *d*. must be given for 40 *l*. Add the 3d part of that quantity to it, the Sum is the total weight that *B* must deliver; and the Value of that 3d part added (which is plainly the 4th part of the whole Sum) shews the Money that *A* has to pay. For *B*'s Barter Price, find it in proportion to the Ready-money Price as *A*'s Price to his Ready-money Price.

CHAP. VII.

Of Tare *and* Tret.

BY *Tare* is commonly meant the *Weight* of the Cask, Chest, or Bag in which Goods are put up, and whose Weight can be known separately from that of the Goods; and which being subtracted from the *gross* Weight (or that of the Cask, &c. and Goods together) the remainder is the Weight of the Goods alone, and is call'd *The Nett Weight*.

But if the *Tare* is not known separately, and an Allowance made for it at so much *per* Hundred weight, or Hundred yards, &c. then the Deduction of the *Tare* is by the *Rule of Three*; which the following Examples will shew.

There is another Allowance made for Dust, Waste, Refuse, or in lack of Goods, call'd *Tret*, which is allow'd and calculated after the same Way.

Exa. 1*st*. At 7 *lb*. *Tare* or *Tret* to 112 *lb*. gross, what is the Tare, and also the Nett Weight, when 746 *lb*. gross was receiv'd? Say, As 112 *lb*. to 7 *lb*. so is 746 *lb*. to the *Tare* sought; which subtracted from 746 *lb*. the Remainder is the Nett weight.

Exa. 2*d*. At 5 *lb*. tret to 112 *lb*. gross, what gross Weight must be receiv'd, when 84 *lb*. Nett was paid for? and, How much is allow'd? Subtract 5 from 112, the Remainder is 107: Then say, As 107 to 112, so 84 to the Gross Weight sought; the Difference of which and 84 is the Allowance. Or thus; As 107 to 5, so is 84 to the Allowance sought; which add to 84, the Sum is the Gross Weight sought.

Thus from the Gross weight, Nett weight, and Allowance, or any two of these in one Case given, with any one of them in another Case, we may find the other two in that other Case.

Observe. There are sometimes two Allowances deducted out of the same Quantity, first *Tare*, and then *Tret*: After the *Tare* is deducted, the Remainder is call'd particularly *Subtle Weight*, out of which the *Tret* is deducted; and the last Remainder is call'd *Nett Weight*.

Exa. 3*d*. *Tare* being allow'd at 4 to 112, and *Tret* at 5 to 112, what is the *Nett Weight* in 87 *lb*. *Gross*?

Say, As 112 to 108 (*viz*. 112 less 4) so is 87 *lb*. to the *Subtle*: Then, as 112 to 107 (*viz*. 112 less 5) so is the *Subtle* to the *Nett*.

And here *Observe*, that if you multiply 108, 107, and 87 continually, also 112 by 112, and divide that Product by this, the Quote is the Nett Weight sought.

CHAP.

CHAP. VIII.

Alligation.

DEFINITION.

ALLIGATION is the *Rule* of mixing ſeveral Simples of the ſame kind, but of different Prices or Qualities, ſo as the Compound may be of a middle Price or Quality: In which there are two principal Caſes, call'd *Alligation Medial* and *Alternate*.

CASE I. *Medial.*

Having the *Rates* (*i. e.* the Price of any quantity of each of ſeveral Simples, or any other Quality by which they are diſtinguiſh'd) and *Quantities* to be mix'd, To find the Rate of the Mixture,

RULE. Find, according to the given Rates, the Value of each given Quantity; then taking the ſum of theſe Quantities, and the ſum of their Values, ſay, If that ſum of Quantities give that ſum of Values, what will any other Quantity give? And you'll find the Rate of the Mixture.

Examples *in which Regard is had to the different Prices of Things.*

1ſt. A Merchant has 13 Gallons of Wine at 17 *s. per* Gal. 11 Gallons at 13 *s.* and 19 Gallons at 14 *s.* If theſe are mix'd, what's the Price of 1 Gallon of the Mixture?

If 1 Gal. gives	17 *s.*	13 Gal. give	221 *s.*			
1 —	15 —	11 —	165			
1 —	14 —	19 —	266			
					s.	*d.*	*f.*
	If	43 —	652 —	1 Gal. gives	15 :	01 :	$3\frac{35}{43}$

If we ſuppoſe 6 Gallons of Water (whoſe Value is nothing) mix'd with theſe, the Proportion is this; If 49 Gallons coſt 652 *s.* what 1 Gallon?

2*d.* A Farmer mixes 7 Bolls of Wheat at 15 *l. per* Boll with 9 Bolls : 3 Buſhels at 10 *l. per* Boll, and 6 Bolls at 12 *l* : 14 *s. per* Boll; What is a Peck of the Mixture worth?

If 1 Boll coſt	15 *l.*	7 Bolls coſt	105 *l.*				
1 —	10 —	9 : 3 Buſh. —	97 : 10 *s.*				
1 —	12 : 14 *s.*	6 : —	76 : 04				
					s.	*d.*	*f.*
	If	22 : 3 —	278 : 14 —	1 Peck gives	15 :	3 :	$3\frac{3}{91}$.

Obſerve: I ſuppoſe here that 4 Pecks make 1 Buſhel, and 4 Buſhel 1 Boll.

Exam-

Examples in which Regard is had not only to the Price, but to the Quality.

1. *Of mixing* M E T A L S.

Observe: An Ounce of *Pure Gold* being reduced into 24 equal Parts, these Parts are call'd *Caracts*; but Gold is often mix'd with some baser Metal, which in the Mixture is call'd the *Alloy*; and according to the proportion of pure Gold which is in every Ounce, so the mixture is said to be so many Caracts fine: Thus; if only 22 Caracts of pure Gold, and 2 of Alloy, it's 22 Caracts fine: If 20 Caracts of pure Gold, and 4 of Alloy, it is 20 Caracts fine: If there is no Alloy, it's 24 Caracts fine, or pure Gold.

Exa. 3d. A Goldsmith mixes 7 Ounces of Gold 23 Caracts fine, with 13 Ounces 19 Caracts fine; What's the Quality of the Mixture? *Answ.* $20\frac{2}{5}$ Caracts.

If 1 *Oz.* has 23 *Car.* of pure Gold, 7 *Oz.* have 161 *Car.*
1 — 19 — 13 — 247

If 20 have 408 — 1 *Oz.* has $20\frac{2}{5}$ *Caracts.*

Suppose there is to be mix'd with these 4 *oz.* of Brass or other Alloy, then add 4 to 20, and the Proportion is, If 24 *Oz.* have 408 *Car.* what 1 *Oz*?

Observe: Silver is valued by the Ounces of pure Silver in a Pound, and 12 Ounces (*Troy* weight) being a Pound, therefore it's call'd 11 or 10, &c. Ounces fine, which has 11 or 10, &c. Ounces pure Silver in the Pound.

2. *Of mixing* M E D I C I N E S *according to their different Degres of* Heat *and* Cold, *or* Dryness *and* Moisture.

Heat and *Cold*, also *Dryness* and *Moisture*, in Medicines, are distinguish'd by different Degrees, thus; There is suppos'd a certain Quality call'd *Temperate*, differing from which there are suppos'd to be 4 degrees of *Heat*, and 4 of *Cold*; also 4 degrees of *Dryness*, and 4 of *Moisture*; all which make 9 degrees of that Quality which regards *Heat* and *Cold*, and as many regarding *Dryness* and *Moisture*, which we may call the Common Quality; so that the 4th degree of *Cold* or *Moist* is call'd the 1st degree of the Common Quality; the 3d degree of *Cold* or *Moist* is the 2d degree of the Common Quality; and so on, as in this

TABLE.

Degrees of the Com. Qual.		
9	4	Degrees of Hot : Dry.
8	3	
7	2	
6	1	
5	0	temperate
4	1	Degrees of Cold : Moi.
3	2	
2	3	
1	4	

Now, if Medicines are to be mix'd with Regard to these Qualities, then the Degrees of the particular Qualities being given, they must be reduced to the Common Quality, and the Operation made with the Numbers of that; as in this *Example.*

Exa. 4th. An Apothecary mixes several Simples, *thus*; 4 ounces cold in the 3d degree, 7 ounces cold in the 1st degree: 5 ounces temperate, and 8 ounces hot in the 4th degree; What is the Quality of the Mixture?

The Qualities given reduced to the common one, are, 2d, 4th, 5th, 9th; then multiplying each Quantity by its degree of the common Quality, the Products are $4 \times 2 = 8 . 7 \times 4 = 28 . 5 \times 5 = 25 . 8 \times 9 = 72$. The Sum of these Products is $8 + 28 + 25 + 72 = 133$; which being divided by the Sum of the Quanti-

ties,

ties, (*viz.* 4 + 7 + 5 + 8 = 24) the Quote is $5\frac{13}{24}$: Which shews the Quality of the Mixture to be betwixt the 5th and 6th degree; *i. e.* betwixt Temperate and the first degree of Heat.

Case 2. *Alternate.*

Having the *Rates* of several Simples to be mix'd, and the *Rate* of the *Mixture*; To find such quantities of the Simples as, being mix'd together, shall bear that common Rate,

Observe: The Mixture Rate must be taken betwixt the highest and lowest Rate of the Simples; else, 'tis plain, the Mixture will not bear that Rate, but will be either of a greater or lesser Rate, as the Simples are either all of a greater or lesser Rate.

Rule. 1°, The Rates being all of (or reduced to) one denomination, and refer'd to Quantities of one denomination, 2°, set the Rates of the Simples in a Column under one another, and the Mixture Rate upon the left hand of these. Then, 3°, connect or link together the several Simple Rates, so that every one less than the Mixture be link'd with some one greater, or with as many as you please that are greater; and every one greater with one less, or with as many lesser as you please. 4°. Take the Difference betwixt the Mixture Rate and that of the several Simples, and write it against all the Simples with which that one (whose Difference it is) is link'd; then, the Sums of the Numbers (of Differences) standing against every Simple Rate, are such quantities of the several Simples, against which they stand, as answer the Question.

Qu. 1. A Merchant would mix Wines at 14 *s.* 19 *s*, 15 *s*, and 22 *s. per* Gallon, so as the Mixture may be worth 18 *s.* What quantity of each may be taken?

$$\begin{array}{c|c|c} & 14 & 1 \\ s. & 15 & 4 \\ 18 & 19 & 4 \\ & 22 & 3 \end{array} \quad \text{or } \begin{array}{c|c|c} & 14 & 4 \\ s. & 15 & 1 \\ 18 & 19 & 3 \\ & 22 & 4 \end{array} \quad \text{or } \begin{array}{c|c|c|c} & 14 & 1:4 & 5 \\ & 15 & 1 & 1 \\ 18 & 19 & 4:3 & 7 \\ & 22 & 4 & 4 \end{array} \quad \text{or } \begin{array}{c|c|c|c} & 14 & 4 & 4 \\ & 15 & 1:4 & 5 \\ 18 & 19 & 3 & 3 \\ & 22 & 4:3 & 7 \end{array}$$

$$\text{or } \begin{array}{c|c|c|c} & 14 & 1:4 & 5 \\ 18 & 15 & 4 & 4 \\ & 19 & 4 & 4 \\ & 22 & 4:3 & 7 \end{array} \quad \text{or } \begin{array}{c|c|c|c} & 14 & 1 & 1 \\ 18 & 15 & 1:4 & 5 \\ & 19 & 4:3 & 7 \\ & 22 & 3 & 3 \end{array} \quad \text{or } \begin{array}{c|c|c|c} & 14 & 1:4 & 5 \\ 18 & 15 & 1:4 & 5 \\ & 19 & 4:3 & 7 \\ & 22 & 4:3 & 7 \end{array}$$

Here the Simples are link'd all the Ways possible, by each of which there is a different Solution, *thus*; In the first Method 14 and 19 are link'd, and 15 with 22: Then the difference betwixt 18 and 14 is 4, which I set against 19; and the difference of 18 and 15 is 3 set against 22: The difference of 18 and 19 is 1 set against 14; and the difference of 18 and 22 is 4 set against 15: And these differences are the Answers, *viz.* 1 Gal. of 14 *s.* Wine, 4 Gal. of 15 *s.* Wine, 4 Gal. of 19 *s.* and 3 Gal. of 22 *s*; which being mix'd together, each Gallon is worth 18 *s.* The same Way understand the 2d Method of linking the Simples. For the 3d Method, 14 *s.* is link'd both with 19 and 22, and 19 both with 14 and 15: Therefore the difference of 18 and 14, which is 4, is set against both 19 and 22; and the difference of 18 and 19 (*viz.* 1) set against both 14 and 15; and thus there come two differences against 14 and 19, which being summ'd, the Answers in this Method are, 5, 1, 7, 4 Gal. of the Simples against which these Numbers stand. From this 'tis easy to understand the other Methods.

The same *Rule* is applicable to mixing *Metals* or *Medicines*, as in the following Question.

Qu. 2.

Qu. 2. To mix Gold 18 Caracts fine, with Gold of 13 Caracts, of 14 Caracts, of 21 and 23, so as the Mixture may be 15 Caracts fine; What Quantities may be taken of each?

		Oz.			Oz.			Oz.			Oz.
	13	3:8		13	6		13	3:6		13	3:6:8
	14	6		14	3:8		14	6:8		14	6
15	18	2	or 15	18	1	or 15	18	2	or 15	18	2
	21	1		21	2		21	2:1		21	2:1
	23	2		23	1		23	1		23	2

These are a Part only of the various Ways of linking the Simples in this Question.

The Reason *of the preceding* Rule.

To *demonstrate* that the preceding *Rule* produces true Answers, I shall,

1°, Suppose only two Simples, as, *Wine* of 15 *s.* a Gallon, and of 22 *s.* to be sold at 18 *s.* the Given Rates stand, according to the Rule, as here; and the Quantities sought are 4 Gallons of the 15 *s.* Wine, and 3 Gallons of the 22 *s.* Wine, which being sold in a mixture at 18 *s.* I say, there is precisely as much gain'd by the one Quantity as is lost by the other; for, each Gallon at 15 *s.* gains 3 *s.* the difference of 15 and 18, and therefore 4 Gallons gain 4 times 3 *s.* Again; each Gallon at 22 *s.* loses 4 *s.* the difference of 18 and 22, therefore 3 Gallons lose 3 times 4 *s.*; but 4 times 3 *s.* is equal to 3 times 4 *s.* therefore the Gain and Loss are equal, and consequently the Quantities mix'd do justly bear the propos'd Rate.

	15	4
18		
	22	3

The same Reason is manifestly good in all Cases of two Simples mix'd according to this Rule, from the Way of placing the Differences alternately against the Simple Rates. Again,

2°. However many Simples there are, and with however many others every one is link'd, since 'tis always a lesser with a greater than the Mixture-price, therefore there is a balance of Gain or Loss upon the Quantities taken from every linking of two Simples; and consequently there must be a balance on the whole: So that the Rule is good in all Cases.

Practical Observations *relating to the preceding Case.*

1st Obs. These Questions of *Alligation Alternate* are of the kind which the Algebraists call *Indeterminate Problems*; *i. e.* which have an infinite number of different Answers; for finding which, their Art gives an universal Rule: But, for the Rule here given, it is limited in its immediate Effect, to the different Answers found by the various Methods of linking the Simples; which can be done only a certain limited number of Ways: *Yet* from this Rule we can find an infinite number of other Solutions. *Thus*;

(1°.) Take any Method of linking the Simples; then take the Quantities arising from that Method; and if you encrease or diminish each of them in the same proportion (*i. e.* by equal *Multiplication* or *Division*); these new Quantities are also true Answers, for that very Reason that they are proportional to those arising immediately from the Linking and Differences; because, if two Quantities of two Simples make a balance of Gain or Loss, with respect to the Mixture-price, so must double or triple, or the half or third part, or any other proportion of these Quantities. And, because these Quantities may

may be encreas'd or diminish'd in an infinite variety of Proportions, therefore 'tis plain that we may proceed to an infinite variety of Solutions.

2°. Or, if we only encrease or diminish the alternate or correspondent Differences of any pair of Simples that are link'd together, or of any two or more pairs, leaving the rest as they are, we may thus also proceed to an infinite number of Solutions.

II *Obs.* Besides the Rates of the Simples and Mixture given, the Question may be limited either to a certain total quantity of the Mixture, or to a certain quantity of some one or more of the Simples.

(1°.) If the Limitation is to a certain total Quantity, then, if the sum of the Quantities found by any one Way of linking the Simples is the given Total, the Question is solv'd: Otherwise (or without trying all the Ways of linking) take any one Method, and raise or diminish the quantity of each Simple found by that linking, in proportion as the given Total is greater or less than that Total found by the linking.

Qu. 3*d*. A Merchant mixes Wines at 14*sh*, 12*sh*, 15*sh*, 18*sh*, and 22*sh*. the Gallon, to be sold at 17*sh*. and would make in the whole 100 Gallons; What Quantity may he take of each?

	12	1	1	$5\frac{15}{17}$
	14	1	1	$5\frac{15}{17}$
17	15	5	5	$29\frac{7}{17}$
	18	5 : 3	8	$47\frac{1}{17}$
	22	2	2	$11\frac{13}{17}$
		Sum	17	100

	12	1 : 5	6
	14	5	5
17	15	5	5
	18	5	5
	22	5 : 3 : 2	10
		Sum	31

If the Given Total were 17 or 31, the Question is solv'd: But, to make 100 gallons, I take either of these Ways of linking, as the first, and say, As 17 Gal. to 100, so is 1 to $5\frac{15}{17}$; so is 5 to $29\frac{7}{17}$, so is 8 to $47\frac{1}{17}$, so is 2 to $11\frac{13}{17}$; which being set against the Correspondent Simples, the Sum makes up 100 Gallons.

After this manner, if you know the Total, and also the Particulars, of any Mixture, you may find how much of each species is in any quantity of the Mixture.

(2°.) If the Limitation is to a certain Quantity of one of the Simples, then, if the same Quantity happen upon that Simple in any one Way of linking, the Question is solv'd; otherwise you must raise or diminish the rest, in proportion as the limited Quantity of that Simple is greater or lesser than the Quantity of it found by the linking.

Thus, in *Qu.* 3*d*, suppose the Mixture ought to have 2 Gallons of the 22 *s*. Wine; then the first Way of linking solves the Question. And if it ought to be 4 Gallons of 22 *s*. say, As 2 Gallons (standing against 22 *s*.) is to 4, so is 1 to 2, so is 5 to 10, and so is 8 to 16: And so the Quantities sought are 2 Gallons of 12 *s*. and also of 14 *s*. 10 Gallons of 15 *s*. and 16 Gallons of 18 *s*.

If the Simple whose Quantity is limited is only once link'd, we need do no more than raise or diminish the Quantity of that one Simple with which it is link'd, and leave the rest as they are. So, in the preceding Supposition, 22 *s*. is only link'd with 15 *s*. and therefore raising 5 Gallons against 15 *s*. to 10 Gallons (which is as 4 to 2) we may take the rest as they stand: For thus there is still a Balance preserv'd in the Gain and Loss.

But if the Simple whose Quantity is limited is link'd with more than one, we may take this Method: Take that part of the Quantity standing against each of these Simples (with which the limited one is link'd) which is the difference of the Mixture Rate and the limited Simple, and raise or diminish it proportionally: The Quantities thus found must be added to the other parts of the Quantities against these other Simples. *Thus*, in *Qu.* 3*d*,

suppose

suppose 'tis requir'd to have 8 Gallons of the 12 *s.* Wine; if the 2d way of linking is chosen, then 12 being join'd with 18, which has but one Difference against it, *viz.* 5, (the Difference of 17 and 12) I say, As 6 to 8, so is 5 to $6\frac{2}{3}$, the Quantity to be taken of the 18 *s.* Wine. Again; 12 being join'd with 22, and the Difference of 17 and 12 being 5, (one of the Differences against 22) I say as before, As 6 to 8, so is 5 to $6\frac{2}{3}$; which being taken instead of 5 against 22 *s.* makes the Total of that Simple $11\frac{2}{3}$ Gall.

(3°.) If the Limitation is to a certain Quantity of more than one Simple, *work thus*; Take these simple Rates, with their given Quantities, and find by *Case* 1. what Rate the Mixture of these by themselves would bear; then take the sum of their Quantities given, with their Mixture-Rate now found, and place that Rate in the Question, instead of the Rates of these Simples; and then the Question is the same as a Limitation to one Simple which is the Total of the Given Quantities now reduced to one Mix'd Rate; by which therefore find the Quantities of the rest, as in the preceding Article, *Thus*; Suppose in *Qu.* 3*d*, that there ought to be 3 Gallons of 12 *s.* Wine, and 7 of 14 *s.* the Mixture Price at which these may be sold is 13 *s* : 4 *d* : and $\frac{4}{5}$; and this being reduced to 5th parts of a Penny, is 804: Therefore the Rates of the other Simples, and also the Mixture, must be reduced to 5th parts of a Penny, and the Question will stand as below; in which, according to the way of Linking chosen, the Quantities are 60 . 300 . 216 and 120; but of the Wines whose Quantities are limited, and whose Mixture Price is 804, (5ths of a Penny) the Quantity ought only to be 10 (*viz.* 3 of the one, and 7 of the other); and therefore the rest are diminish'd accordingly to 50, 36. and 20; which are true Answers to the Question.

	804	60	10
1020	900	300	50
	1080	216	36
	1320	120	20

Take Notice, That if the Mixture Rate of the Simples limited is such that the Given Mixture Rate is not a Medium, when that other Mixture Rate is placed as a Simple, then the Limitation makes the Question impossible.

III *Obs.* If a Mixture is made of several Simples whose Rates are known; with the Rate of the Mixture, and total Quantity mix'd, we may find how such a total Quantity might be mix'd of these Simples to bear the Given Rate, by the 1st Article of the 2d *Observation.* But it is to be observ'd, that the Mixture has perhaps been made after another manner. So, in *Qu.* 1, the 1st and 2d Ways of Linking make the same total Quantity.

IV *Obs.* When the Quantity of one Simple is limited, if that Simple is an exact *Medium* betwixt some other two (exceeding the one as much as it wants of the other) then, having link'd the Simples any one way, if the limited Quantity is less than what is found by the linking, take the half of what it is less, and add so much to each of these two Simples betwixt which it is an exact Medium; for thus the total Quantity found by the linking is preserv'd, since what is taken less in one, is made up out of others; and what is so taken less and more than the Quantities found by the linking, are of equal Value, because the middle Price is an exact Medium betwixt the other two; Therefore the Rate of the Mixture is never alter'd.

Again; If the Limitation is greater than what is found by the linking, take the half of what it exceeds from the Quantities of each of those betwixt which the Price whose Quantity is given is an exact Medium; but if this half is greater than these Quantities, you must take another Method.

Thus, in *Qu.* 3d, let the Limitation be to 2 Gallons of the 15 *s.* Wine, then take the first way of linking, in which the Quantity of that Wine is 5 Gallons; then, because 15 *s.* is an exact Medium to 12 and 18, I take the Difference of 2 and 5, which is 3, and its half

half $1\frac{1}{2}$ I add to 1 and 8 the Quantities againſt 12 and 18, the Sums are $2\frac{1}{2}$ and $9\frac{1}{2}$: The reſt of the Quantities ſtand as they are

But if the Limitation is to 8 Gallons of the 15 *s.* Wine, which exceeds 5 by 3, then, becauſe $1\frac{1}{2}$ is greater than 1 againſt 12, therefore I cannot take the Method preſcrib'd, with this Way of Linking, but with the ſecond Way it can be done. And if it could not be done with either of theſe Linkings, we muſt either find one in which it can be done, or ſolve the Queſtion by the General Rule in Article (2°.) *Obſ.* II.

V *Obſ.* From the Method explain'd in the preceding *Obſervation*, it's plain how we may, in ſome Circumſtances, limit both the total Quantity and ſome one of the Simples, *thus*; if the Simple which is limited is an exact Medium betwixt other two, then take any one Linking, and proportion the Quantities to the Total limited; then apply the Method of the laſt *Obſervation*, if poſſible.

VI *Obſ.* In Mixtures one Ingredient may be ſuch, as to bear no Value in the Mixture, but only to encreaſe the Quantity, and diminiſh the Value; Therefore let its Rate be repreſented by 0, as Water mix'd with Wine; Braſs, or other Alloy, mix'd with Gold and Silver.

Exa. 1. If 8 Gal. of Wine at 9 *s. per* Gallon, 12 Gal. at 8 *s.* are mix'd together; How much Water muſt be added to make the Mixture worth only 6 *s. per* Gallon? I find the Mixture Rate of the 8 Gal. and 12 Gal. then I take 20 Gal. at that Rate to mix with Water whoſe Rate is 0: Which is done by the Method of *Article* 2d, *Obſ.* II.

Exa. 2. A Goldſmith would mix Gold, 18 Caracts fine, 20 Caracts fine, 24 Caracts, and a quantity of Alloy to make the Mixture 19 Caracts fine; How much may be taken of each? Repreſent the Rate of the Alloy by 0, and proceed as in *Caſe 2d.*

VII *Obſ.* Beſides the mixture of Liquors, or any other kind of Things, the ſame Rules are applicable where *Perſons* are the Subjects, *thus:*

Exa. 1. 8 Men being boarded at the rate of 6 *l.* a Quarter for every Man, 6 Women at 5 *l.* for each Woman, and 4 Children at 2 *l.* for each Child; How much does each Perſon pay a Quarter, taking them at an equal rate, one with another? This is plainly a Queſtion of *Caſe* 1, and to be ſolv'd after that manner.

Exa. 2. If the Quarter's Board for a Man is 5 *l*, for a Woman 4 *l*, for a Child 3 *l*, and for a Servant 1 *l.* how many Men, Women, Children, and Servants may be taken to board, ſo as their Board, at an equal Rate, may come to 3 *l*: 5 *s.* for each Perſon? This is a plain Queſtion of *Caſe* 2d.

I add the following Queſtions *for a further Exerciſe.*

Qu. 4*th.* There is a Mixture of 40 Gallons of Wine worth 10 *s.* a Gallon, part of which is at 8 *s*, part at 9 *s*, at 12 *s*, and 14 *s*. What ſhall be added to it, to make the Mixture worth only 11 *s*?

To ſolve this Queſtion, The Simple to be added muſt be of ſuch a Price that the new Mixture Price lie betwixt it and the former; therefore, if there is not ſuch a Simple among theſe which are already in the Mixture, another to anſwer the Queſtion muſt be brought in, and the Solution is thus made: Take the 40 Gallons at 10 *s*, and find how much of any Simple of greater value than 11 *s.* (becauſe this is greater than 10 *s*) muſt be mix'd with theſe 40 Gallons to bear 11 *s.* in Mixture: Which is done by the Method of *Article* 2d, *Obſ.* II. preceding.

Qu. 5th A Mixture being made thus; 14 *lb.* weight of Sugar at 7 *d. per lb*, 16 *lb.* at 9 *d.* and 30 *lb.* at 10 *d.* How much, and of what kind, may be added, that in every *lb.* of the Mixture there be 6 *oz.* of the Sugar at 10 *d*, without changing the quantity of the other kinds in every *lb*? and, What Rate will the Mixture bear?

Firſt, find how much of the 10 *d.* Sugar is in 1 *lb.* of the Mixture, as it ſtands already; then (1) if that is more than 6 *oz.* (as in this caſe 'tis 8 *oz.*) ſubtract 6 *oz.* from it, and multiply the difference (2) by the total number of *lb*'s (60), the Product is 120 *oz.* Now, if we could take out 120 *oz.* of the 10 *d.* Sugar, and in its place put in 120 *oz.* of any other kind than what is already in the Mixture, it's plain the Conditions of the Queſtion would be anſwer'd: Therefore, in the firſt place, we ſhall add 120 *oz.* of ſome other kind, as if as much of the 10 *d.* Sugar were actually taken out; but becauſe it is not ſo, we muſt add to it as much of all the other kinds (including that new kind ſuppos'd to be already added) as ſhall make each *lb* (of this ſum) have as much of each kind as are in each *lb* of the other 60 *lb*; which is done *thus:* Find how many times 6 *oz.* are contain'd in 120, [and if there were a Remainder, as here there is none, I add that Remainder of the 10 *d.* Sugar to the Mixture, (*i. e.* to the 120) whereby there will be no Remainder, and the Quote will be 1 more than it was with the Remainder] this Quote ſhews how many *lb.* are to be added to the given total Mixture (60 *lb.*) that each may have 6 *oz.* of 10 *d.* Sugar; which are to be made up *thus:* Firſt, there is the 120 *oz.* of the 10 *d.* Sugar ſuppos'd at firſt to be taken out (for as much of a new kind put in) and now as 'twere put back again, together with the Remainder laſt mention'd (where there is any) then you muſt find how much of each of the other Simples [including that new kind whereof there is ſuppos'd to be already added 120 *oz.* for as much of 10 *d.* Sugar ſuppos'd to be taken out] there is in each of the preceding 60 *lb.* Theſe Quantities muſt be multiply'd by the Difference of 60 *lb.* and the new total Quantity or Mixture; the Products are what muſt be taken of each of theſe other Simples (beſides the 120 *oz.* already ſuppos'd to be added of the new kind) and thus the Conditions of the Queſtion are anſwer'd.

For the *Rate* of this New Mixture, it's found by *Caſe* 1.

But, Secondly, if there is in the Mixture leſs than 6 *oz.* of 10 *d.* Sugar to every *lb.* (as ſuppoſe there be only 4 *oz.*) then the difference is 2 *oz.* which multiply'd by 60, makes 120 *oz.* or 7 *lb*: 8 *oz.* Take double of this, with as much as is already in the Mixture of 10 *d.* Sugar (*viz.* 30 *lb.*) the Sum is 45 *lb.* to be added of 10 *d.* Sugar. Then I conſider how much of each of the other kinds is in the Mixture, and from the total of theſe I take 120 *oz.* (or 7 *lb*: 8 *oz.*) in any manner, *i. e.* all out of one kind, if poſſible, or part of one and part of another, till the whole is ſubtracted, marking what hereby remains in each kind; then as much of each kind as theſe Remainders being alſo added to the Mixture, will bring the Total (which is now 120 *oz.* more than double of what it was before) to ſuch a Mixture as ſhall have 6 *oz.* of 10 *d.* Sugar to every *lb*, and the ſame quantities of each of the reſt as were at firſt to every *lb.*

Qu. 6th. There is a Mixture made of Wheat 6 *s*, 7 *s*, and 8 *s.* the Buſhel; the total Quantity is worth 100 *l.* and the proportion of the particular Quantities are thus; For every 2 Buſh. of 6 *s.* there are 3 of 7 *s*; and for every 3 of 6 *s.* there are 4 of 8 *s.* How much of each kind is in the Mixture? and, What is the Rate of the Mixture?

Firſt, I reduce the Proportions of the Quantities to a Series of three Numbers, *thus:*

Sh. 6	2	:	3	6
7	3			9
8			4	8

As 2 buſh. (of 6 *s.*) to 3 buſh. (of 6 *s.*) ſo is 3 buſh. (of 7 *s.*) to a 4th, which is $4\frac{1}{2}$ buſh. (of 7 *s.*) ſo that for every 3 buſh. of 6 *s.* there are $4\frac{1}{2}$ of 7 *s.* and 4 buſh. of 8 *s.* And multiplying all by 2, to make 'em Integers, the proportional Quantities of each are 6 . 9 . 8. Next I find the Mixture Price from theſe Quantities; and then, by that Rate, I find

I find how much the 100 *l.* will buy; and then I divide that Quantity into three parts proportional to *6.9.8*.

Qu. 7th. A Mixture is to be made of Wine at 18 *s. per* Gallon, at 16 *s.* and 9 *s.* How much may be taken of each to make 48 Gallons that shall be worth in all 28 *l*: 16 *s*?

Find what 1 Gallon of the Mixture is worth, *viz.* 12 *s.* then find what Quantities may be taken of the several Simples, to make a Mixture at 12 *s.* and the total 48 Gallons.

Observe: By this Method you can always have at least one Solution to all Questions of this kind (*viz.* wherein the Rate of each Simple, the total Mixture, and total Value are given) provided the Question is possible; as 'tis not if the Mixture Rate found be not betwixt the highest and lowest of the Simples; and provided also that the Nature of the Subject does not limit the Answer to Integral Numbers, as when the Numbers sought are *Animals*, (see the following *Example*) for then, tho' the Question be possible, and hath several Solutions, yet none of them may be found by this Method, because of Fractions brought by it into the Answers: And this Method will have the same Defect in any Subject, if we limit the Answer to Integers.

For *Example*; Apply the Numbers of the preceding Question to another Subject, *thus*: 48 Persons, Men, Women, and Children, pay all together at a Feast 28 *l*: 16 *s.* whereof every Man pays 18 *s.* every Woman 16 *s* and every Child 9 *s.* How many were of each? By the preceding Method we shall find 9 Men, 9 Women, and 30 Children; which satisfie the Question: Tho' it has also another Solution (which cannot be found by this Method) *viz.* 2 Men, 18 Women, and 28 Children.

Exa. 2d. 15 Birds cost in all 5 *s.* whereof there were Partridges at 7 *d.* Quails at 5 *d.* and Larks at 2 *d.* How many were of each? If we try this by the preceding Rule, we can find no Answer, because of Fractions; and indeed it has but one Solution in Integers, which is found to be 3 Partridges, 5 Quails, and 7 Larks.

Qu. 8th. A Mixture was made of 10 Gal. Wine, 4 Gal. Brandy, and 12 Gal. Water. Out of the whole was drawn 8 Gal. and as much Water put in to fill it up: Then was drawn out 10 Gal. after which was put in 6 Gal. of Wine: Again there was drawn out 5 Gal. and 3 Gal. of Brandy put in. How much Wine, Brandy, and Water is at last in the Mixture?

Find how much of each Species is in the 8 Gallons first drawn out, by which you'll know what remains of each: To which is added a Gallon of Water; then find how much of each is contain'd in the 10 Gall. next drawn out, whereby you'll know what remains of each: To which is added 6 Gall. Wine; then find how much of each is in the 5 Gall. drawn out, and you'll know how much remains of each; to which 3 Gall. of Brandy is to be added.

CHAP.

CHAP. IX.

Of Exchange.

THE *Reduction* of different Coins, or any Denominations of Money (whether they have Real Coins answering to them, or not) from one to another, *i. e.* finding how many of one Species or Denomination are equal in Value to a given Number of another; with other Questions relating to the same Subject, is what I here call *Exchange.*

Observe I. If the Question is about the Reduction of such Species as are aliquot Parts one of another, as Pounds, Shillings, and Pence, the Work is only simple *Multiplication* or *Division*; as has been shewn in *Book* 1, *Ch.* 7, § 4 & 5; but, in all other Cases, there is a variety of Applications of the *Rule of Three.*

II. To reduce any number of one Species to another, there must always be known the Proportion betwixt the two Species, either immediately, or by the immediate Proportions betwixt each of them and one or more other Species.

III. When we know any two Numbers of different Species that are of equal Value, these two Numbers express the Proportions, or Rate of *Exchange*, of these Species, whether they belong both to one Country, or to different Countries, [as, if we suppose 3 Guineas of *Britain* equal to 3 *l*: 3 *s.* or 5 *s.* Sterling equal to 3 Guilders *Amsterdam.*] But, in Common Acceptation, when we speak of *Exchange*, it's understood of the Species of different Countries; and the *Rate* of *Exchange* is express'd by so much of the one Place equal to an *Unit* of a certain Species of another: So, if 5 *s.* Sterling are equal to 3 Guild. *Amsterdam*, tho' this does really express the Proportion, yet more commonly we say, The *Rate of Exchange* is at 1 *l.* Sterl. to 12 Guild. (found by the given Proportion; thus, if 5 *s.* give 3 Guild. in what 20 *s.*) or as 1 Gu. to 1 *s*: 8 *d.* Sterling.

IV. The Values of the Coins and Monies of different Countries; in what Specie's Denominations they exchange with one another, with the Limits of the Common *Rates* of *Exchange*, (for they vary often) are Matters of Fact which belong to another province: My Business here is, to teach the Calculation, or shew how the Rules of *Arithmetick* are apply'd to this Useful Subject: And, to do this to the best advantage, I shall give more suitable Numbers of Questions, and distinguish them under such Heads as comprehend Things of most ordinary Occurrence; and then add a few other Questions, that are less common, that from the Whole you may be Master of what can happen upon this Subject, that depends upon *Common Arithmetick.*

§ I.

§ I. QUESTIONS *relating simply to the* Par *of* Exchange, *or finding the Number of one Species equal to a Given Number of another; the Proportion being given directly, or by the mediation of other Species.*

(1°) If the Proportion is given betwixt two Species directly, the Solution is by one Operation of the *Rule of Three.*

Quest. 1. A Merchant at *Amsterdam* paid 150 Guilders for 13*l* : 15 *s.* Sterling, receiv'd by his Correspondent at *London* ; What is the *Rate of Exchange ?* or, What is 1 Guilder valued at in English Money ? Say, If 150 Guild. give 13 *l* : 15 *s.* what will 1 Guilder ? *Answ.* 22 *d.* Or, if it's ask'd what 1 *l.* Sterling is valued at in Dutch Money, say, If 13 *l* : 15 *s.* give 150 Guild. what 1 *l ?*

Qu. 2d. A Merchant at *London* receiv'd 100 *l.* Sterling, for the value paid by his Correspondent at *Paris* at the rate of 3 *s* : 6 *d.* Sterl. *per* Crown ; How many Crowns were paid at *Paris ?* Say, If 3 *s* : 6 *d.* give 1 Cr. what 400 *l* ?

(2°) When the Proportion is given between each of the two Species in question, and a third Species, the Solution is by two Operations of the *Rule of Three.*

Qu. 3d. If I would exchange 200 Ducats, worth 7 *s.* a piece, for Dollars at 4 *s* : 8 *d.* a piece, how many Dollars ought I to have ? Say, If 1 Ducat give 7 *s.* what 200 ? *Answ.* 1400 *s.* Then, if 4 *s* : 8 *d.* give 1 Dollar, how many will 1400 *s* ?

Or thus : If 4 *s* : 8 *d.* give 1 Dol. how many will 7 *s* ? *Answ.* $1\frac{1}{2}$; so that 1 Ducat is worth $1\frac{1}{2}$ Doll. Then say, If 1 Duc. give $1\frac{1}{2}$ Dol. how many will 200 give ?

Observe : Had it only been requir'd to find the direct Proportion betwixt Crowns and Dollars, then say, If 7 *s.* give 1 Ducat, how many will 4 *s* : 8 *d* ? *Answ.* $\frac{2}{3}$ of a Ducat. Or say, If 4 *s* : 8 *d.* give 1 Dol. how many will 7 *s* ? *Answ.* $1\frac{1}{2}$ Doll.

Qu. 4th. Exchange from *London* to *Amsterdam* being at 1 *l.* Sterling for 34 *s.* Flem. and from *Amsterdam* to *Paris* at 5 *s.* Flemish for 1 Crown, what is the *Exchange* betwixt *London* and *Paris*, according to that Course? *Answ.* $6\frac{4}{5}$ Crowns for 1 *l.* Sterling, or 2 *s* : $11\frac{5}{17}$ *d.* for 1 Crown ; found thus : If 5 *s.* give 1 Cr. what 34 *s* ? It is $6\frac{4}{5}$ Cro. which is the Value of 1 *l.* because 34 *s.* is equal to 1 *l.* Or say, If 34 *s.* give 1 *l.* what 5 *s* ? It is 2 *s.* : $11\frac{5}{17}$ *d.*

Lond.	*Amst.*	*Par.*
1 *l.* =	34 *s.*	
	5 *s.* =	1 Cr.

Observe : If 'tis demanded to find the Value of 500 Cr. in English Money, according to that Course of Exchange ; then, having found the Rate of Exchange, say, If 1 Cro. give 2 *s.* $11\frac{5}{17}$ *d.* what 500 Crowns ? or, If $6\frac{4}{5}$ Cr. give 1 *l.* what 500 Crowns?

If a Sum English (as 100 *l.*) is given to find its Value in Crowns, 'tis only stating the Terms otherwise, according to the *Rule of Three*, thus ; If 1 *l.* give $6\frac{4}{5}$ Cr. how many will 100 *l.* give ?

(3°) When the Proportion is given betwixt one Specie and another, betwixt this other and a third, betwixt this third and a fourth, and so on as far as you please, to find the Exchange betwixt the first and last Species ; the Solution is by one fewer Operations of the *Rule of Three* than there are different Species, in the manner of the following Question.

Qu. 5*th.*

Qu. 5*th*. Exchange betwixt *London* and *Amsterdam* being at 1 *l.* Sterling for 38 *s.* Flemish; betwixt *Amsterdam* and *Frankfort* at 6 *s.* Flemish for 65 Cruitzers; betwixt *Frankfort* and *Paris* at 56 Cruitzers for 1 Crown; what is the Exchange, according to that Course, betwixt *London* and *Paris*? *Answ.* 1 *l.* for $7\frac{20}{27}$ Crowns; found thus: Set down the Given Terms as in the Margin, and work from the left Hand to the right, *thus*; say, If 6 *s.* give 66 Cru. what will 38 *s*? The Answer is 418 Cruit. which is set under *Frankf.* Then say, If 54 Cru. give 1 Crown, what 418 Cru.? The Answer is $7\frac{20}{27}$ Cr. which is set under *Paris.* And thus the Exchange betwixt *London* and *Paris* is found to be at 1 *l.* for $7\frac{20}{27}$ Crowns.

Lond.		*Amst.*		*Frankf.*		*Paris*
1 *l.*	=	38 *s.*				
		6 *s.*	=	66 Cru.		
				54 Cru.	=	1 Cr.
1	=	38 *s.*	=	418 Cru.	=	$7\frac{20}{27}$ Cr.

Observe, (1°) However many different Places are propos'd, you go thro' them all in the same manner.

(2°) The undermost Line shews not only the Exchange betwixt the first and last Places, but also betwixt any two of them; the Quantities in that Line being evidently all equal in Value, from the Nature of the Operation.

(3°) If there are two different given Species for one Place, they must be reduced to one Species: So, if the Exchange betwixt *Amsterdam* and *Frankfort* were express'd by 36 Stivers for 66 Cruitzers, then we must first reduce the 38 *s.* to Stivers, or the 36 Stivers to Shillings, by the known relation of *Stivers* and *Shillings*; which is 6 Stivers to 1 Shilling. Or, if it were 36 Stivers for 1 Florin, then you must also reduce the 1 Florin to Cruitzers, or the 54 Cruitzers to Florins. And if this Reduction cannot be done, *i. e.* if the relation of these Species is not known, the Question cannot be solv'd.

(4°) If there's another Given Quantity of the first and last Place, to find a Quantity of equal Value in the other, it's a plain Application of the *Rule of Three*, from the *Rate* of Exchange found (as was observ'd upon the preceding Question).

Some Contractions of the preceding RULE *explain'd.*

(5°) All these Operations of the *Rule of Theee* may be reduced to one Division, *thus*; Multiply the *Consequents* of all the *Proportions* [*i. e.* the Numbers upon the right hand, or the first Number under every Place] continually into one another; also all the *Antecedents*, except the first, [*i. e.* the Numbers upon the left hand, or second Number under every Place] make the first Product *Dividend*, and the second *Divisor*, and the *Quote* is the Number sought of the Species of the last Place, equal to the Number under the first Place; *thus*, in the preceding Example, 38 multiply'd by 66, produces 2508, (the number of Crowns being 1, does not multiply) then 54 by 6, produces 324; and 2508 divided by 324, produces $7\frac{20}{27}$.

The *Reason* of this will be manifest, by considering how the several Operations of the *Rule of Three* are made; for the Answer of the first Operation is the Quote of 38 multiply'd by 66, and the Product divided by 6; which we may express thus, $\frac{38 \times 66}{6}$; then for the second Operation it is, the last Answer multiply'd by 1, (which is still only $\frac{38 \times 66}{6}$) and the Product divided by 54, which is $\frac{38 \times 66}{6 \times 54}$; according to the Direction now given: And how many Places soever there be, the Reason is manifestly the same.

Again;

Again; If any *Divisor* and *Dividend* can be divided by any Number without a Remainder, then if we take the two Quotes, and divide the one of them by the other, there will arise the same Quote as from the given *Dividend* and *Divisor*; therefore this Work may be much contracted, *thus*; If among the Numbers that produce the Divisor there are any the same with what are among those that produce the Dividend, cast them out of both (*i. e.* do not use them in the Multiplication): And if any two Numbers, one belonging to the Divisor, and one to the Dividend, can each be divided without a Remainder by any the same Number; Take these Quotes in place of those given Numbers, in forming your *Divisor* and *Dividend*. *So*, suppose in the preceding Example it were 1 *l.* English for 54 *s.* Flemish, then this 54 (which belongs to the Numbers that form the Dividend) and the 54 Cruitzers (which belong to the Numbers that form the Divisor) may both be cast out. And, as the Question is already put, tho' there is no Number the same, yet for 6 and 66 we may take 1 and 11, the Quotes of 6 and 66 divided by 6; also for 38 and 54 we may take 19, 27, the halfs of the former, and so the Dividend is $19 \times 11 = 209$, the Divisor is 27, and the Quote $7\frac{20}{27}$.

§ II. QUESTIONS *wherein* Gain *and* Loss, *and the Allowances to* Factors, *are consider'd.*

Qu. 6th. *A* of *London* draws upon *B* of *Amsterdam* 500 Guilders at 22 *d. per* Guild. for which *B* redraws upon *A* at 21 *d. per* Guilder, with Provision at $\frac{1}{2}$ *per Cent.* and 10 Guilders of Brokerage; How much will *A* pay? and, Whether has he gain'd or lost? *Ans.* He pays $11018\frac{3}{4}$ *d.* and loses $18\frac{3}{4}$: found thus; As 1 Guil. to 22 *d.* so is 500 Guil. to 11000 *d.* receiv'd by *A* for the Draught; then, as 100 to $\frac{1}{2}$, so 500 to $2\frac{1}{2}$, the Provision due to *B*; which, with 10 Guild. Brokerage, added to 500, makes $512\frac{1}{2}$, for which he draws on *A*; therefore say, As 1 Guild. to $21\frac{1}{2}$ *d.* so $512\frac{1}{2}$ to $11018\frac{3}{4}$ *d.* which *A* pays for the redraught; so that he loses $18\frac{3}{4}$ *d.*

Observe, (1°) If *A* had remitted to *B*, with Orders to remit the Value back again; then having found what *B* receives by *A*'s Remittance, from that subtract *B*'s Provision and Brokerage, the Remainder is what he remits to *A*; whose Value being found, the comparison of that and what *A* paid for the Remittance shews what he gains or loses.

(2°) If *A* draws upon *B*, and afterwards remits the Value to him, he must add the Provision due to *B*, to the Sum which *B* paid, and remit the Total.

(3°) If *B* by *A*'s Order draw upon him, and afterwards remits him the Value, then *B* deducts his Provision and double Brokerage from what he receiv'd by the Draught, and remits the Remainder.

(4°) In all the Cases where *A* draws or remits, you must also consider what Brokerage it may have cost him, in order to know his Gain or Loss.

Qu. 7th. If Exchange from *London* to *Amsterdam* is at 1 *s*: 10 *d.* for 1 Guilder, and to *Paris* at 3 *s*: 8 *d.* for 1 Crown; also from *Amsterdam* to *Paris* at 40 Stivers for 1 Crown; whether is it most profitable that *London* remit directly to *Paris*, or by way of *Amsterdam* [*i. e.* remit to *Amsterdam*, to be remitted from that to *Paris*]? Find what 1 *l.* Sterling is worth at *Paris*, according to the Course with *Amsterdam* (by Question 4th) and also according to the *Exchange* directly with *Paris*; the Comparison of these Values of 1 *l.* gives the Answer.

Observe, 1, what Allowance is due to the Factor at *Amsterdam* is to be deduced from the Money he receives (*i. e.* the value of 1 *l.* Sterling in Guilders) and the Remainder is what he remits to *Paris*.

2. If there are more than 3 Places in the Question, and the Exchange is given betwixt one and another in a series, as in *Quest.* 5*th*, and also betwixt the first and last; to find which is most profitable, to Remit from the first to the last Place directly, or thro' all these Places; then you must find what is the Exchange betwixt the first and last Place according to the Courses thro' all the other Places, and compare that with the given Exchange betwixt the first and last Place.

Qu. 8*th*. A Merchant at *London* remits to *Amsterdam*, at the rate of 18 *d.* Sterling *per* Guilder: His Correspondent at *Amsterdam* remits the same by Order to *Bourdeaux*, at 3 Guilders *per* Crown, rebating $\frac{1}{3}$ *per Cent.* for his Provision: How much will be receiv'd at *Bourdeaux* for 10 *l.* Sterling paid at *London*? *Answ.* $44\frac{8}{27}$ Crowns, *thus*, As 18 *d.* to 1 Guilder, so is 10 *l.* to $133\frac{1}{3}$ Guilders: Then, as 100 to $\frac{1}{3}$, so is $133\frac{1}{3}$ to $\frac{4}{9}$, the Provision; which taken from $133\frac{1}{3}$, there remain $132\frac{8}{9}$. Then, as 3 Guild. to 1 Crown so is $132\frac{8}{9}$ to $44\frac{8}{27}$.

Again: Suppose 'twere ask'd what was paid at *London* when 500 Crowns were receiv'd at *Paris*? To find this say, As 1 Cro. to 3 Guild. so 500 Cro. to 1500 Guild. which were paid at *Amsterdam* for the Remittance to *Paris*: Then, $\frac{1}{3}$ from 100, there remain $99\frac{2}{3}$; and as $99\frac{2}{3}$ to 100, so is 1500 to $1505\frac{5}{299}$, the Guilders receiv'd at *Amsterdam* by the Remittance from *London*. Lastly, As 1 Guild. to 18 *d.* so is $1505\frac{5}{299}$ Guild. to $27090\frac{90}{299}$ *d.* the Engl. Money sought. Or thus, As $99\frac{2}{3}$ to 100, so is 500 Cr. to $501\frac{201}{299}$, the Crowns that would have been got at *Paris* had no Allowance been deduced at *Amsterdam*: Therefore say, As 1 Cro. to 3 Guild. so $501\frac{201}{299}$ to $1505\frac{5}{299}$, the Guild. receiv'd at *Amsterdam*. By which find the English Money; as before.

Again, let us suppose the French and English Money are both known, as, 430 Cro. and 100 *l*, Sterling; To find the Allowance *per Cent.* which the Factor at *Amsterdam* has; say, As 18 *d.* to 1 Guild. so 100 *l.* to $1333\frac{1}{3}$ Guild. which was receiv'd at *Amsterdam*. Then find how many Crowns this is worth at 3 Guild. *per* Crown; if the number were 430, then there was no Allowance deduced at *Amsterdam*, but it is $444\frac{4}{9}$; from which take 430, the Remainder is $14\frac{4}{9}$: Then say, As $444\frac{4}{9}$ to $14\frac{4}{9}$, so is 100 to the Allowance upon 100.

Qu. 9*th*. Suppose *all* as in the preceding Question, with this further, That the Merchant at *London* draws upon *Bourdeaux* for the Crowns receiv'd there at 50 *d. per* Crown, paying 5 *s.* Brokerage; allowing also $\frac{1}{2}$ *per Cent.* to his Correspondent at *Paris*; What does he gain or lose by this Negotiation? *Answ.* He loses 1 *l* : 1 *s* : $4\frac{7}{27}$ *d.* which is discover'd thus; As 100 to $\frac{1}{2}$, so is $44\frac{8}{27}$ Crowns (receiv'd at *Paris*) to $\frac{299}{1350}$, the Allowance; which taken from $44\frac{8}{27}$, leaves $44\frac{101}{1350}$: which being drawn upon *Paris* at 50 *d. per* Crown, there is receiv'd at *London* for it $2203\frac{20}{27}$ *d.* or 9 *l* : 3 *s* : $7\frac{20}{27}$ *d*; from which subtract the Brokerage 5 *s.* the Remainder is 8 *l* : 18 *s* : $7\frac{20}{27}$ *d.* which the Merchant at *London* had clear for the Draught upon *Paris*: But he paid 10 *l.* for the Remittance; therefore he has lost 1 *l* : 01 *s* : $4\frac{7}{27}$ *d.*

Observe: If the Correspondent at *Bourdeaux* remits the Value in his hands to *London*, then, from the Sum receiv'd by the Remittance from *Amsterdam*, he deduces his Provision and Brokerage, and remits the Remainder to *London*.

Qu. 10*th*.

Qu. 10*th*. A Rix-dollar is worth at *Amsterdam* 50 Sols, and at *Coningsberg* 90 Gros; the Exchange betwixt *Coningsberg* and *Amsterdam* is at 230 Gros for 6 Florins (equal to 120 Sols); whether is it most profitable that *Amsterdam* remit to *Coningsberg* in Specie, or by Exchange? *Answer.* 'Tis best to remit by Exchange: Which is discover'd thus; As 50 *s.* to 90 Gros, so 120 *s.* to 216 Gros, which must be remitted in Specie, (*i. e.* its Value in Rix-dollars) but by the Exchange there will be 230 Gros for 120 *s.* therefore 'tis best to remit by Exchange; the Difference is, 14 Gros will be had more for every 6 Florins.

Observe: If *Amsterdam* is to remit to *Coningsberg* 1000 Gros, and you would know which to chuse, and what is sav'd upon the whole; find what 1000 Gros will cost, both in Specie and by Exchange, and then you'll have the Difference, *thus*; As 90 Gros to 50 *s.* so 1000 to 555 *s.* which must be paid in Specie. *Again*; As 230 Gros to 120 *s.* so 1000 Gros to 521$\frac{17}{23}$ *s.* to be paid by Exchange.

Qu. 11*th*. *Amsterdam* draws upon *Rouan* 400 Crowns at 87 *d.* Flemish *per* Crown, for which *Rouan* redraws upon *Amsterdam* at 90 *d.* with $\frac{1}{2}$ *per Cent.* for Provision; what has *Amsterdam* gain'd or lost? Say, As 1 Crown to 87 *d.* so 400 Crowns to 34800 *d.* receiv'd by Draught. *Again*, adding 2 for Provision to 400 Crowns, say, As 1 Crown to 90 *d.* so 402 to 36180 paid for the Redraught; so that *Amsterdam* has lost 1380 *d.*

Qu. 12*th*. *Amsterdam* remits to *Paris* 1000 Crowns at 78 *d. per* Crown, paying for Brokerage 540 *d.* which *Paris* remits to *Amsterdam* at 80 *d. per* Crown, rebating $\frac{1}{2}$ *per Cent.* for Provision; What is gain'd or lost? Say, As 1 Crown to 78 *d.* so 1000 Crowns to 78000; to which add 540, the Sum is 78540 *d.* given out. Again, subtract 5, the Provision, from 1000 Crowns, remains 995; then, As 1 Crown to 80 *d.* so 995 to 79600 *d.* receiv'd: So there is gain'd 1060 *d.*

Qu. 13*th*. If Exchange betwixt *Midleburg* and *London* is at 35 Sols for 1 *l.* Sterling, also betwixt *Midleburg* and *Amsterdam* at 1$\frac{1}{2}$ *per Cent.* of Advance for *Midleburg*, (*i. e.* 101$\frac{1}{2}$ *s.* at *Amsterdam*, worth 100 *s.* at *Midleburg*) at what Rate ought *Amsterdam* to remit to *London*, to receive the Return by *Midleburg* at the 'foresaid Course, and gain 5 *per Cent.*? Find the Course betwixt *London* and *Amsterdam*, after the manner of Question 4*th*, it is 1 *l.* for 35$\frac{21}{40}$ *s.* Then, to remit to *London* with 5 *per Cent.* Gain, he must pay less than the Course, in proportion as 100 is less than 105; therefore say, As 105 to 100, so 35$\frac{21}{40}$ to 33$\frac{35}{42}$, which is the Rate at which he ought to remit to *London*. Or, you may find this Answer at once, *thus*; As 105 to 101$\frac{1}{2}$, so 35 to 33$\frac{35}{42}$; the Reason of which you'll find easily, by comparing the two former Proportions, *viz.* 100 : 101$\frac{1}{2}$: : 35 : 35$\frac{21}{40}$ and 105 : 100 : : 35$\frac{21}{40}$: 33$\frac{35}{42}$; where the two middle Terms of the one being the two Extremes of the other, and the Products of Extremes and Means being equal from the nature of Proportion, it follows that $101\frac{1}{2} \times 35 = 105 \times 33\frac{35}{42}$; and hence $105 : 101\frac{1}{2} :: 35 : 33\frac{35}{42}$.

§ III. QUESTIONS *relating to what is call'd* The Arbitration *of* Exchange.

Qu. 14*th*. *A* of *Rochel* orders *B* of *Amsterdam* to draw upon him at $97\frac{3}{4}$ Sols for 1 Crown, and to remit the same to *Hamburg* at 34 Sols for 1 Dollar. *B* cannot draw, but at 97 Sols for 1 Crown; How ought he to remit to follow his Order? *Answ.* At $33\frac{17}{23}$ Sols for 1 Dollar: Found *thus*; As $97\frac{3}{4}$ to 97, so 34 to $33\frac{17}{23}$; for the Course being below the Order in the Draught to *Rochel*, it ought to be so proportionally in the Remittance to *Hamburg*. Or, the Reason of the Work may be conceiv'd *thus*: If for 1 Cro. which *A* pays at *Rochel* he gets only 97 Sols at *Amsterdam* by the Course, instead of $97\frac{3}{4}$ which was his Order, then for 1 Dollar he receives at *Hamburg* he ought to pay proportionally fewer Sols than 34, which was his Order.

Roch.	*Amst.*	*Hamb.*
Cr.	Sols.	
1 =	$97\frac{3}{4}$	Doll.
	34 =	1
1 =	97	

Qu. 15*th*. *A* of *Amsterdam* orders *B* of *Paris* to draw upon him at $99\frac{1}{2}$ Den. for 1 Crown, and to remit the same to *London* at $49\frac{3}{4}$ *d.* Sterling for 1 Crown. *B* can remit at $50\frac{1}{2}$ *d.* for 1 Crown; How ought he to draw to follow his Order? *Answ.* At 101 Den. for 1 Cro. Found *thus*; As $49\frac{3}{4}$ to $50\frac{1}{2}$, so is $99\frac{1}{2}$ to 101; for *A* receiving more than his Order at *London*, for 1 Crown paid at *Paris*, he ought to pay proportionally more than his Order at *Amsterdam*, for 1 Crown receiv'd at *Paris*.

Amst.	*Par.*	*Lond.*
Den.	Cr.	d.
$99\frac{1}{2}$ =	1 =	$49\frac{3}{4}$
	1 =	$50\frac{1}{2}$

Qu. 16*th*. *A* of *Coningsberg* orders B of *Amsterdam* to remit to *Rouan* at $103\frac{1}{2}$ *d.* Flemish for 1 Crown, and to draw the same on him at 225 Gross for 1 *l.* Flemish, *B* cannot draw under 230 Gross; How ought he to remit to follow his Order? *Answ.* At $101\frac{1}{4}$ *d.* Flem. for 1 Cro. Found thus: As 230 to 225, so is $103\frac{1}{2}$ to $101\frac{1}{4}$. For if *A* pays at *Coningsberg* more Gros than Order (for 1 *l.* receiv'd at *Amsterdam*) he ought to receive proportionally more Crowns than Order at *Rouan* (for $103\frac{1}{2}$ *d.* paid at *Amsterdam*) or, which is the same thing, he ought to pay proportionally less than Order at *Amsterdam*, for 1 Crown receiv'd at *Rouan*.

Con.	*Amst.*	*Rou.*
	d.	Cro.
Gro.	$103\frac{1}{2}$ =	1
225 =	1 *l.*	
230 =	1 *l.*	

Qu. 17*th*. *A* of *Cologn* orders *B* of *Amsterdam* to remit to *Dantzick* at 1 *l.* Flemish for 230 Gros, and to draw upon him at 100 for $102\frac{1}{2}$ to be paid at *Cologn*; but *B* remits at 1 *l.* for 228 Gros, and draws at 100 for 102; Has he follow'd his Order? and, if not, is *A* Gainer or Loser? *Answ.* He has not follow'd his Order, and *A* is Loser: Which is discover'd *thus*; As 230 Gross to 228, so is $102\frac{1}{2}$ to $101\frac{14}{23}$, which is the Number to be paid at *Cologn* for 100 receiv'd at *Amsterdam*, to make the Course and Order proportional in Drawing and Remitting; for if *A* gets less than Order at *Dantzick*, he ought to pay less than Order at *Cologn*, but by the Course he pays 102, therefore is a Loser.

Col.	*Amst.*	*Dant.*
	1 *l.* =	230 gr.
$102\frac{1}{2}$ =	100	
	1 *l.* =	228 gr.
102 =	100	

Qu. 18*th*

Qu. 18*th.* *A* of *Amsterdam* orders *B* of *London* to draw upon *Rouan* at 35 *d.* Sterling *per* Crown, and to remit the same to him at 1 *l.* Sterling for 35 Sols. But *B* draws at $34\frac{1}{2}$ *d. per* Crown, and remits the same at 1 *l.* for $36\frac{1}{2}$ Sols: Has he follow'd the Order? and, if not, is *A* Gainer or Loser? *Answ.* He has not follow'd his Order, and *A* is Gainer: Which is discover'd *thus*; As $36\frac{1}{2}$ Sols to 35, so is 35 *d.* to $33\frac{41}{73}$, the Number to be receiv'd at *London* for 1 Crown paid at *Rouan*, to make the Course and Order proportional in Drawing and Remitting; for it's plain, that if *A* receives at *Amsterdam* more Sols than Order, for 1 *l.* paid at *London*, he ought to pay at *Rouan* proportionally more Crowns than Order (for 35 *d.* receiv'd at *London*); or, which is the same thing, to receive proportionally less at *London*, for 1 Crown paid at *Rouan*; but he receives $34\frac{1}{2}$ *d.* and therefore is Gainer.

Amst.	*Lond.*	*Rou.*
	35 *d.* =	1 Cr.
35 Sols =	1 *l.*	
	$34\frac{1}{2}$ *d.* =	1 Cr.
$36\frac{1}{2}$ =	1 *l.*	

Qu. 19*th.* *A* of *Rouan* orders *B* of *Amsterdam* to draw upon him at 97 Den. for 1 Crown, or upon *London* at 35 Skilings for 1 *l.* Sterling. According to the Course, *B* can draw upon *Rouan* at 98 Den. for 1 Crown; and upon *London* at $35\frac{1}{2}$ Sk. for 1 *l.* Sterling: Which of them ought *B* to chuse to serve his Employer best? *Answ.* He ought to draw upon *London*: Which is thus discover'd; If *A* is willing to pay 1 Crown at *Rouan* for 97 *d.* receiv'd at *Amsterdam*, or to pay 1 *l.* at *London* for 35 Sk. receiv'd at *Amsterdam*; then 'tis plain, that to follow his Order, if *B* receives 98 *d.* instead of 97 (for 1 Crown) he ought to receive proportionally more than 35 Sk. (for 1 *l.*) And that proportional Number is $35\frac{35}{97}$; (for, As 97 to 98, so is 35 to $35\frac{35}{97}$) which is less than $35\frac{1}{2}$ receiv'd by the Course, so that the Course from *Amsterdam* to *London* exceeds the Order more in proportion than from *Amsterdam* to *Rouan*; and *A* will have more Money in proportion lying at *Amsterdam*, for the same Sum paid at *London*, than if the Draught were upon *Rouan*.

Rou.	*Amst.*	*Lond.*
1 Cro. =	97 *d.*	
	35 Sk. =	1 *l.*
1 Cro. =	98 *d.*	
	$35\frac{1}{2}$ Sk. =	1 *l.*

Qu. 20*th.* *A* of *Dantzick* Orders *B* of *Amsterdam* to remit to him, at 1 *l.* Flemish for 220 Gros; or to *Hamburg* at $33\frac{1}{2}$ Skilings for 1 Dollar; but the Course is at 1 *l.* for 218 Gros at *Dantzick*, and 34 Skilings for 1 Dollar at *Dantzick*; Which ought *B* to chuse to serve his Employer best? *Answ.* He ought to remit to *Dantzick*: Which is discover'd *thus*; Since *A* is willing to pay 1 *l.* at *Amsterdam* for 220 Gros at *Dantzick*, or to pay $33\frac{1}{2}$ Sk. at *Amsterdam* for 1 Dollar at *Hamburgh*; therefore if he can get only 218 Gros for 1 *l.* he ought to have proportionally more than 1 Dollar at *Hamburgh* for 34 Skil. at *Amsterdam*; or, which is the same thing, he ought to pay less than 34 Sk. for 1 Dollar: Therefore say, As 220 Gros to 218, so is $33\frac{1}{2}$ Sh. to $33\frac{43}{220}$, which would be paid at *Amsterdam*, for 1 Dollar at *Hamburgh*, if the Course and Order were proportional to both Places; but by the Course *A* must pay 34 Sk. which is more than he would pay in proportion of the Order and Course to *Dantzick*; Therefore it's best that *B* remit to *Dantzick*.

Dantz.	*Amst.*	*Ham.*
220 gr. =	1 *l.*	
	$33\frac{1}{2}$ Sk. =	1 dol.
218 gr. =	1 *l.*	
	34 Sk. =	1 dol.

Observa-

Observation *relating to the last* 7 Questions.

There's one General Method may be taken with all these Questions, which is this: Reduce all the Given Proportions to such Numbers, as that those under the middle Place be the same in all (as 'tis in *Quest.* 15*th*); and you may also chuse that Number what you please; and then, from the Numbers under the first and last Place, you will easily find the Answer. *Thus*, in *Quest.* 14*th* say, As $97\frac{3}{4}$ is to 1, so is 34 to $\frac{136}{391}$. So that the Exchange betwixt *Rochel* and *Amsterdam* is reduced to 34 Sols for $\frac{136}{391}$ Crowns. Again say, As 97 to 1, so is 34 to $\frac{34}{97}$; the Question will stand as in the Margin. Then say, As $1\frac{36}{391}$ to $\frac{34}{97}$, so 1 Dol. to $2\frac{3}{194}$ Dollars: So the Remittance to *Hamburgh* ought to be at 34 Sols for $1\frac{3}{388}$ Dollars; which is the same Proportion as found the other Way, only in different Numbers. The Reason of the Work will be in some Cases clearer by this Method; but the Work often more tedious.

Roch.	*Amst.*	*Hamb.*
$\frac{136}{391}$ Cr. $=$ 34 *s.*		
	34 $=$	1 Dol.
$\frac{34}{97} = 34$		

§ IV. *Containing a few* Questions *of another kind than any of the preceding, for a farther Exercise upon this Subject.*

Qu. 21*st.* A Merchant would exchange 200 *l.* Sterling for Dollars or Crowns: He is offer'd Dollars at 4 *s*: 6 *d.* which are worth but 4 *s*: 3 *d.* or Crowns at 5 *s.* worth but 4 *s*: 8 *d.* Which of them shall he take to lose least? and, How many will he receive? Find how many Dollars at 4 *s*: 6 *d.* and Crowns at 5 *s.* he would get for 200 *l.* then find the Value of that number of Dollars at 4 *s*: 3 *d.* and that number of Crowns at 4 *s*: 8 *d.* the Comparison of these Values will shew which is of greatest Value; and the Value of that which is the greatest, compar'd with 200 *l*, shews what he loses.

Qu. 22*d.* A Merchant at *Amsterdam* drew Bill upon *London* for 300 *l.* Sterling, receiving the Value in Crowns at 4 *s*: 6 *d.* and Dollars at 4 *s.* and got an equal number of each; What is that number? Add 4 *s*: 6 *d.* to 4 *s.* and say, If the Sum 8 *s*: 6 *d.* buy 1 of each Species, how many times 1 of each Species will 300 *l.* buy? The *Answer* is, $705\frac{90}{102}$, (for, dividing 300 *l.* by 8 *s*: 6 *d.* the whole is 705, and 90 remains) so he receiv'd 705 Dollars, and as many Crowns, with $\frac{90}{102}$ parts of 1 of each. And because there is a Remainder in the Division, therefore this shews that the Exchange cannot be made exactly, by a certain number of Crowns and Dollars; so that 90 *d.* remaining, the 300 *l.* is 90 *d.* better than the Sum of 705 Dollars, and 705 Crowns: Wherefore he who receives only 705 Dollars and 705 Crowns, must give 90 *d.* less than 300 *l.*

Again; If the Proportion of the number of Crowns and Dollars is suppos'd to be any other than Equality (for example, 2 Dollars for every 3 Crowns) then add the Value of 2 Dollars and 3 Crowns, and divide by that Sum: The Quote shews how many times 2 Dollars and 3 Crowns are to be receiv'd; and if there is a Remainder, 'tis to be consider'd as so many Units of the Denominator of the Divisor; and so much the Dividend is of more Value than the number of Crowns and Dollars found.

If

If there are more than two different Species, as, *Crowns*, *Dollars*, *Ducats*, *Pistoles*, the manner of working is the same; for, if an equal number of each is suppos'd, then add the value of an Unit of each, and by that Sum divide: If their numbers are not equal, then either (1°) the correspondent Numbers of each that are equal Numbers of Times taken is given, as, for every 2 Dollars 3 Ducats, 5 Crowns, and 1 Pistole: And here we add the valueof 2 Dollars, 3 Ducats, 5 Crowns, and 1 Pistole, and by that Sum divide. (2°) If the Proportion of the Numbers are given, but not in one Series, as, suppose for 3 Dollars 2 Ducats, for 3 Ducats 4 Crowns, and for 7 Crowns 1 Pistole; then we must reduce these Proportions to one Series of correspondent Numbers of each, *thus*; Keep the first two Numbers, *viz.* 3 Dollars 2 Ducats, then find how many Crowns for 2 Ducats (at 3 Ducats for 4 Crowns) and how many Pistoles for that number of Crowns last found (at the rate of 1 Pistole to 7 Crowns) then proceed as before, by adding the Values of these correspondent Numbers of the different Species; and, to go thro' the reduction of the Proportions more orderly, set the Species and their proportional Numbers down as here:

Doll.	:	Duc.	:	Crowns	:	Pistoles.
3	:	2				
		3	:	4		
				7	:	1

Qu. 23*d.* If I receive 11 Crowns and 7 Dollars for 4 *l* : 10 *s* : 10 *d.* or 4 Crowns and 3 Dollars for 1 *l* : 15 *s.* the Value of 1 Crown and 1 Dollar being the same in both, What is that Value?

This Question may be solv'd two Ways; (1°) Reduce the Money all to Pence; then, to make the same number of Dollars in both Cases, multiply the 1090 *d.* and its equivalent number of Crowns and Dollars by 3, also the 420 *d.* and its equivalent number of Crowns and Dollars by 7, the Products must still be of equal value: And if the one Line of Products be taken from the other, it's manifest that the Remainders will also be equal; and, because the Dollars are equal in both, therefore there are none in the Remainders; and so we have found that 5 Crowns are equal to 330 *d.* consequently 1 Crown is 5 *s* : 6 *d.* Then, to find the Value of 1 Dollar, multiply 5 *s* : 6 *d.* by 4, the Product is 22 *s.* the Value of 4 Crowns; which taken from 35 *s.* the Value of 4 Crowns and 3 Dollars, there remains 13 *s.* the Value of 3 Dollars; wherefore 4 *s* : 4 *d.* is the Value of 1 Dollar.

				Cr.		Dol.
4 *l* : 10 *s* : 10 *d.*	or	1090 *d.*	=	11	+	7
1 *l* : 15 *s.*	or	420 *d.*	=	4	+	3
		3270 *d.*	=	33	+	21
		2940 *d.*	=	28	+	21
		330 *d.*	=	5 Cro.		

(2°) We may also solve it *thus*; say, If 1090 *d.* buy 18 Pieces (*viz.* 11 Crowns and 7 Dollars) how many will 420 *d.* buy? The Answer is, $6\frac{102}{109}$: Then I divide this Number into Crowns and Dollars, in the same proportion as 18 is to 11 Crowns and 7 Dollars, *thus*; As 18 to 11 Crowns, so $6\frac{102}{109}$ to $4\frac{26}{109}$ Crowns: And this taken from $6\frac{102}{109}$ there remains $2\frac{76}{109}$ Dollars. And, because the same 420 *d.* buy 4 Crowns and 3 Dollars,

therefore

$4\frac{26}{109}$ Cr. $+ 2\frac{76}{109}$ Dol. $= 4$ Cr. $+ 3$ Dol.

$\frac{26}{109}$ Cr. $= \frac{33}{109}$ Dol. or 26 Cr. $= 33$ Dol.

therefore $4\frac{26}{109}$ Crowns and $2\frac{76}{109}$ Dollars are of equal Value with 4 Crowns and 3 Dollars: And if we cast equal numbers of the same Species out of both sides, the Remainders will still be equal: So cast out 4 Crowns and $2\frac{76}{109}$ Dollars, the Remainders are $\frac{26}{109}$ Crowns, equal to $\frac{33}{109}$ Dollars; and consequently 26 Crowns equal to 33 Dollars. Then say, If 33 Dollars are worth 26 Crowns, how many Crowns are 3 Dollars worth? It is $2\frac{4}{11}$; then, Consequently, 4 Crowns and $2\frac{4}{11}$ Crowns (which are worth 3 Dollars) are worth 420 *d.* because 4 Crowns and 3 Dollars are worth 420 *d.* Lastly, If $6\frac{4}{11}$ Crowns are worth 420 *d*, 1 Crown is worth 66 *d.* or 5 *s* : 6 *d.* By which find the Value of the Dollar; as before.

Of the Reduction *of* Weights *and* Measures.

The *Reduction* of *Weights* and *Measures* of different Places is done after the same manner as Money and Coins. The Proportions being known either immediately or mediately, thro' several different Places, therefore I shall give only two Examples.

Qu. 1st. If 1 Eln of *Amsterdam* is equal to $1\frac{1}{5}$ of *London*, how many Elns of *Amsterdam* are in 1000 *English* Elns? Say, If $1\frac{1}{5}$ give 1, what will 1000? It is $833\frac{1}{3}$.

Qu. 2d. If 3 *lb* weight at *A* are equal to 2 *lb* at *B*, and 5 *lb* at *B* equal to 2 *lb* at *C*, and 7 *lb* at *C* equal to 8 *lb* at *D*; What is the Proportion betwixt *A* and *D*?

A	*B*	*C*	*D*
3 =	2		
	5 =	2	
		7 =	8
3 =	2 =	$\frac{4}{5}$ =	$\frac{32}{35}$

Set down the Names and Numbers given, as in the Margin; and work as directed in *Qu.* 5*th*, whereby you will find not only the Proportion of the first and last Places, but of all the Places to one another: so here 3 *lb* at *A* is equal to 2 *lb* at *B*, $\frac{4}{5}$ at *C*, and $\frac{32}{35}$ at *D*.

CHAP.

CHAP. X.

Of Intereſt *and* Annuities.

§. 1. *Of* Intereſt.

DEFIN. 1. INTEREST is the Præmium or Money paid for the Loan or Uſe of Money; and is diſtinguiſh'd into two Kinds, *Simple* and *Compound.*

2. *Simple Intereſt* is that which is paid for the *Principal,* or Sum lent, at a certain Rate or Allowance made by Law, (or Agreement of Parties) whereby ſo much as 5 *l.* or 6 *l.* or any other Sum, is paid for 100 *l.* lent out for one Year; and more or leſs proportionally for greater or leſſer Sums; and for more or leſs time. For *Example*: If it's 6 *l.* to 100 for one Year, it's 3 *l.* for half a Year, and 12 *l.* for two Years. Alſo 12 *l.* for one Year of 200 *l.* and 6 *l.* for half a Year; and ſo on for other Sums and Times.

3. *Compound Intereſt* is that which is paid for any principal Sum, and the ſimple Intereſt due upon it for any time, accumulated into one principal Sum. *Example:* If 100 *l.* is lent out for one Year at 6 *l.* and if at the End of that Year the 6 *l.* due of Intereſt be added to the Principal; and the Sum 106 *l.* conſider'd as a new Principal bearing Intereſt for the next Year, (or whatever leſs time it remains unpaid) this is called *Compound Intereſt,* becauſe there is Intereſt upon Intereſt, which may go on, by adding this ſecond Year's Intereſt of 106 *l.* to the Principal 106 *l.* and making the Whole a Principal for the next Year.

SCHOLIUM. Our Law allows only *Simple Intereſt*: But abſtracting from the Reaſon of the Law, [which may be the encouraging of Trade, by employing Money that way rather than upon Intereſt] if taking Intereſt be at all juſt, *Compound Intereſt* cannot be unreaſonable. For if I can demand my Intereſt when it is due, I may take that Intereſt Money, and lend it out again upon Intereſt to any other Perſon; why then may I not lend it out alſo to the Perſon who has my principal Sum? And, in point of Right and Juſtice, it is the ſame thing if I continue or leave that Intereſt in his Hands: there is the ſame Reaſon that it ſhould bear Intereſt after it becomes due, as that the original Sum ſhould do ſo.

PART 1. *Of* Simple Intereſt.

We have already ſeen in the Rule of Five, how, from any ſuppoſed principal Sum, with the ſuppoſed Intereſt of it for any ſuppoſed time, we can find at that rate, or upon that Suppoſition, the Intereſt of any other principal Sum for any time; or the Principal correſponding to any Sum of Intereſt and Time; or laſtly, the Time in which any Principal gives any Intereſt.

We ſhall now conſider the Application of that Rule more particularly; by limiting the Queſtions to the more common Circumſtances of Buſineſs. Thus: As the Law, or Agreement of Parties, fixes a certain Ratio, or, as we call it, Rate of Intereſt, which is ſo much on the 100 *l.* for one Year; from this we can eaſily find the proportional Intereſt on 1 *l.* for one Year, being plainly the $\frac{1}{100}$ Part of the Intereſt of 100 *l*; ſo if this is 5 *l.* that is .05 *l.*; if this is 6 *l.* that is .06 *l.*; and if this is 5 *l.* 10 *s.* or 5.5 *l.* that is .055 *l.* Wherefore

fore if we underſtand the Rate of Intereſt to be the Intereſt of 1 *l.* for one Year, the more common Queſtions about ſimple Intereſt will relate to theſe four things, *viz.* any principal Sum, its Intereſt, the Time in which it gives that Intereſt, and the Rate (or Intereſt of 1 *l.* for one Year) according to which that Principal, Intereſt and Time are adjuſted to one another.

From which we have four *Problems*: In the Rules whereof, I ſuppoſe the Principal and Intereſt expreſſed in the Denomination of Pounds, by reducing what is leſs than 1 *l.* to a Decimal of 1 *l.* and the Time to be expreſſed in Years, and decimal Parts of one Year.

PROBLEM 1. Having any principal Sum, and Time, with the Rate of Intereſt given, to find the Intereſt of that Sum for that Time and Rate.

Rule. Multiply the Principal, Rate, and Time continually into one another, the Product is the Intereſt ſought.

Obſerve; If we expreſs the Principal by p, the Intereſt by n, the Time by t, and the Rate by r; then this Rule is thus repreſented, $n=ptr$.

Example. The Rate of Intereſt being .05 *l.* what is the Intereſt of 85 *l.* for 4 Years and 3 quarters, or 4.75 Years? Anſwer, 20 *l.* 3 *s.* 9 *d.* $=20.1875\,l.=85\times4.75\times.05$.

DEMONSTR. If we ſtate this Queſtion by the Rule of Five, it ſtands thus: If 1 *l.* in 1 y^r gives .05 *l.* what 85 *l.* in 4.75 y^r, and the two firſt Terms are the Diviſors in the two ſimple Proportions; but theſe being both Units, the Anſwer is the Product of the other three. The Reaſon is the ſame in all Caſes, by putting r, p, and t in place of theſe particular Numbers. Or we ſhall repeat the Reaſoning thus: If 1 *l.* give r, p will give pr in the ſame time. Again; if p in one Year give rp, in t Years it muſt give $t\times rp$ or trp.

PROB. 2. Having the Rate, Principal and Intereſt to find the Time.

Rule. Divide the Intereſt by the Product of the Rate and Principal, the Quote is the Time; thus, $t=\frac{n}{rp}$.

Example. The Rate .05 *l.* Principal 85 *l.* Intereſt 20 *l.* 3 *s.* 9 *d.* or 20.1875 *l.* The Time is 4.75 Years, or 4 Years and $\frac{3}{4}$. Thus; $4.75=\frac{20.1875}{85\times.05}$ or $\frac{20.1875}{4.25}$.

DEMONSTR. This Rule is deduced from the former; thus, Since $n=trp$; then dividing both Sides by rp, it is $\frac{n}{rp}=t$. Or it may be deduced from the Rule of Five, as the former.

PROB. 3. Having the Principal, Intereſt, and Time, to find the Rate.

Rule. Divide the Intereſt by the Product of Principal and Time, the Quote is the Rate. Thus $\frac{n}{tp}=r$.

Example. $n=20.1875\,l.$ $t=4.75\,y^r.$ $p=85\,l.$ then is $r=.05\,l.=\frac{20.1875}{4.75\times85}$ or $\frac{20.1875}{403.75}$.

DEMONSTR. Since $n=trp$, divide both by tp; it is $\frac{n}{tp}=r$.

PROB. 4. Having the Rate, Time and Intereſt, to find the Principal.

Rule. Divide the Intereſt by the Product of Rate and Time, the Quote is the Principal; thus, $\frac{n}{tr}=p$.

Exam.

Exam. $n = 20.1875\ l.$ $t = 4.75$ yr. $r = .05\ l.$; then is $p = 85\ l. = \frac{20.1875}{4.75 \times .05}$ or $\frac{20.1875}{.2375}$.

DEMONSTR. Since $n = trp$, divide both by tr, the Quotes are $\frac{n}{tr} = p$.

SCHOLIUM. If the Intereſt of any Sum for any time is added to the Principal, this Total or Sum is called the *Amount*, (*viz.* of the Principal and its Intereſt for that time.) And then from theſe four things, *viz.* the *Amount*, (which we ſhall call a) the Principal, the Time and Rate, ariſe other four *Problems*; for each of theſe may be found from the other three. Thus:

PROB. 5. Having the Principal, Time and Rate, to find the Amount.

Rule. Find the Intereſt by *Prob.* 1. Add it to the Principal, the Sum is the Amount. Thus, by *Prob.* 1. the Intereſt is rtp; therefore the Amount is $a = rtp + p$. The Reaſon is evident.

And obſerve, Becauſe $rtp = rt \times p$, and $p = 1 \times p$; therefore $rtp + p = \overline{rt + 1} \times p = a$. And ſo the Rule may be expreſſed thus: To the Product of the Rate and Time add Unity; and multiply the Sum by the Principal, the Product is the Amount.

Example. What is the Amount of $246\ l.$ Principal in 2 Years and $\frac{1}{2}$, or 2.5 yr, the Rate of Intereſt being .05 $l.$? Anſwer, $246\ l. + 30.75\ l. = 276\ l.$ 15 $s.$ for the Intereſt is $= 246 \times .05 \times 2.5 = 30.75\ l.$ Or thus; $.05 \times 2.5 = .125\ l.$ to which add 1, it is $1 + .125\ l.$ which multiplied by 246, produces $276.75\ l.$

PROB. 6. Given the Principal, Amount and Time, to find the Rate.

Rule. Take the Difference betwixt the Principal and Amount, and divide it by the Product of the Time and Principal, the Quote is the Rate. Thus, $r = \frac{a - p}{tp}$.

Exam. Suppoſe $a = 276.75\ l.$ $p = 246$, $t = 2.5$ yr; then is $r = .05\ l. = \frac{276.75 - 246}{2.5 \times 246} = \frac{30.75}{615}$.

DEMONSTR. Since by *Prob.* 5. $a = trp + p$; take p from both Sides, it is $a - p = trp$; then divide both by tp, it is $\frac{a - p}{tp} = r$.

Or we may deduce it thus; $a - p$ is the Intereſt of p for the Time t, and Rate r; then by the Rule of Five, find what is the Intereſt of it for one Year, when the Principal p gives $a - p$ in t Years; the State of which is thus, $p . t . \overline{a - p} .. 1 . 1$. By that Rule it is $\frac{a - p}{tr}$; for the 4th and 5th Terms are Units, and the middle Term is to be multiplied into theſe two, whoſe Product is nothing but $\overline{a - p}$.

PROB. 7. Given the Amount, Principal and Rate, to find the Time.

Rule. Take the Difference of the Amount and Principal, and divide it by the Product of the Principal and Rate, the Quote is the Time. Thus, $t = \frac{a - p}{rp}$.

Exam. Suppoſe $a = 276.75\ l.$ $p = 246$. $r = .05$; then is $t = 2.5$ yr. $= \frac{276.75 - 246}{246 \times .05} = \frac{30.75}{12.3}$.

DEMONSTR. In the last we saw $a-p=trp$. Divide both by rp, it is $\frac{a-p}{rp}=t$. Or by the Rule of Five, thus, If 1 *l.* give r in 1 yr. in what Time will p give $\overline{a-p}$, following the common Rule, the Time sought is $\frac{a-p}{rp}$ as before.

PROB. 8. Given the Amount, Rate and Time, to find the Principal.

Rule. Add 1 to the Product of the Rate and Time, and by that Sum divide the Amount, the Quote is the Principal; thus, $p=\frac{a}{rt+1}$.

Exam. $a=276.75$ *l.* $r=.05$ *l.* $t=2.5$ yr.; then is $p=246=\frac{276.75}{2.5\times .05+1}=\frac{276.75}{1.125}$.

DEMONSTR. By *Prob.* 5. it is $a=\overline{rt+1}\times p$; divide both by $\overline{rt+1}$, it is $\frac{a}{rt+1}=p$. Or from the Rules of Proportion; thus, Find the Interest of 1 *l.* for the Time given, at the given Rate, it is rt for 1 yr.$:r::t:rt$; then find a Principal so proportioned to a, as 1 *l.* is to $1+rt$; thus, $1+rt:1::a:\frac{a}{1+rt}$.

SCHOLIUM. *Concerning the* Rebate *or* Discompt *to be allowed for paying of Money before it falls due.*

IN the last *Problem*, we have the Foundation of the Rule for *Discompt*, or the Allowance to be made for the paying of Money before it falls due; and which is supposed to bear no Interest till after it is due: for in that Case there is no Reason for Discompt, as there is in the other; which is this, That the Debtor can employ his Money upon Interest, (or as he pleases) till the time of Payment comes; and to pay it before it is due, is to communicate that Benefit to the Creditor, who ought therefore to pay for it; and the Question is, What Discompt or Allowance is to be made? The Debtor will be apt to argue in this manner: By paying the Debt before it falls due, I lose the Interest I could make of it till then, and therefore all that Interest must be discompted. But this Argument is false: For tho' it be true, that from this time to the time the Debt falls due, he would make so much Interest; yet it is not true, that by paying the whole Debt without Discompt, he loses so much at this time as that Interest; because he can't be said to have lost it, till the time come at which he should receive it; therefore he can be said to lose no more at this time than such a Sum, as being laid out upon Interest from this time, till the time of Payment of the Debt would amount to the Interest of the Debt for the same time; therefore such a Sum being found by *Prob.* 8. is the Discompt. In order to which, find the Amount of 1 *l.* for the given time; then say, If that Amount give 1 *l.* what will the Interest of the Debt for the time give.

This I shall also confirm by another Rule for finding the Discompt; which is this: Find by *Prob.* 8. a Principal, which being laid out at a certain Rate of Interest, for so long as a Debt is paid before it was really due, will then amount to that Debt; that Sum is the thing which truly satisfies or clears the Debt, and so is called the present Worth of it; the Difference betwixt this and the Debt being the true *Discompt*. The Reason is this; If the Money paid is such, that being laid out at Interest from the time it is paid to the time the Debt falls due, would amount to the Debt, then neither Party is wronged; for if he who receives it, or he who pays it, lays it but upon Interest so long, he will have then as much to receive as the Debt. For *Example*: If Interest is at 5 *per Cent.* 100 *l.* laid out for one Year

Year is worth 105 *l.* at the Year's End; therefore 105 *l.* due at a Year's End is worth only 100 *l.* presently paid, and the Discompt is 5 *l.* which is one Year's Interest of 100 *l.* not of 105 *l.*

Now I shall demonstrate, that both the Rules given for finding of *Discompt* are in effect the same. Thus: The Interest of 1 *l.* for the time t (*viz.* the time of the Discompt) at the Rate r is rt, therefore the Amount of 1 *l.* for that time is $1+rt$; and the Principal corresponding to rpt (the Interest of p for the same time) in the same Proportion as 1 to $1+rt$ is $\frac{rpt}{1+rt}$ for $1+rt : 1 :: rpt : \frac{rpt}{1+rt}$; which is the Discompt of the Debt by the first Rule. Again; the Principal corresponding to p (the Debt) in the same Proportion as $1+rt$ to 1, is $\frac{p}{1+rt}$, for $1+rt : 1 :: p : \frac{p}{1+rt}$; which is the true Payment to be made; which subtracted from p, the Remainder is $p - \frac{p}{1+rt} = \frac{prt}{1+rt}$, the Discompt by the former Rule.

To sum up this, let p be any Debt payable after the time t, and r the Rate of Interest; then

The RULES *for* Present Worth *and* Discompt *are*

1. *Present-Worth* $= \frac{p}{1+rt}$

i.e. Divide the Debt by the Sum of 1, and the Product of the Rate and Time.

Discompt $= \frac{prt}{1+rt}$

i.e. Divide the continual Product of the Debt, Rate and Time, by the Sum of 1, and the Product of Rate and Time.

Observe, The Discompt upon any Debt for one Year being found, (which will be $\frac{pr}{1+r}$, for t being here one, has no Effect). Dr. *Harris* teaches to find the Discompt of the same Debt for any Part of one Year, in proportion to the Time; so the Discompt for $\frac{1}{2}$ a Year is $\frac{1}{2}$ of the Discompt for one Year. But the Error of this Rule will appear by comparing it with the preceding general one, thus; One Year's Discompt is $\frac{pr}{1+r}$; Let t be any Time less than 1 Year, expressed fractionally, then the Doctor's Rnle is $1 : \frac{pr}{1+r} :: t : \frac{prt}{1+r}$ = the Discompt sought: Whereas by the Rule above demonstrated, it is $\frac{prt}{1+rt}$: Which will be greater than the other if t is a Fraction; because then $1+rt$ is less than $1+r$; but if t is a whole or mixt Number, $1+rt$ is greater than $1+r$; whence $\frac{prt}{1+rt}$ is less than $\frac{prt}{1+r}$.

PART II. *Of* Compound Interest.

THE Calculation of *Compound Interest* supposes a certain stated Time for which Interest is at a certain determinate Rate; after which that Interest becomes a Principal bearing Interest. For *Example*: If Simple Interest is payable yearly, so is the Compound; and if Simple Interest is payable Quarterly and Monthly, so is the Compound: But then for the Interest of any Time less than that to which the Rate is determined, there are different Opinions about the Way of calculating it, which I shall explain in the following *Problem*; where I shall first consider the Rate of Interest as determined to one Year, and the

the Time of the Question limited to whole Years, and afterwards consider other Suppositions about the Rate and Time.

Observe, again; That the finding of the whole Improvement of Compound Interest depends upon the Rule for finding the Amount; therefore the Questions in which the Amount is concerned must be explained before these in which the Interest alone is concerned, contrary to what we did in Simple Interest.

In the following *Problems*, by the *Rate* is understood the Amount of 1 *l* and 1 Year's Simple Interest. So Interest being at 5 *per Cent.* the Rate or Amount of it is 1.05 *l*. which I shall mark R, the Time t being whole Years.

PROBL. I Given the Principal, Rate, and Time, to find the Amount.

Rule. Find such a Power of the Rate whose Index is the Time (*i.e.* multiply the Rate by it self, and this Product by the Rate, and so on, multiplying the last Product by the Rate, till the Number of Multiplications be equal to the Time less 1; so for 2 Years multiply once; for 3 Years multiply twice, *&c.*) This Power of the Rate is the Amount of 1 *l.* for the Time and Rate given; which, multiplied by the given Principal, the Product is the Amount sought.

Universally, $A = p \times R^t$: And *observe*, that the Logarithms will be convenient for finding R^t, if t is a great Number.

Example: What is the Amount of 160 *l* at Compound Interest for 4 Years, the Rate being 1.06 *l.* (*i.e.* 6 *per Cent*)? *Answer*: 201.9963136 *l*; which is 201 *l.*: 19 *sh.* 11 *d.* and a Fraction less than 1 Farthing. Found thus:

$R = 1.06$. and $R^t = \overline{1.06}^4$ or $1.06 \times 1.06 \times 1.06 \times 1.06 = 1.26247696$, then $p \times R^t = 160 \times 1.26247696 = 201.9963136$.

DEMONSTR. If 1 *l* amount in 1 Year to R, then all this forborn another Year, the Amount is $R \times R$ or R^2; for all principal Sums have necessarily the same Proportion to their Amounts for the same Time and Rate, and $1 : R :: R : R^2$, which is therefore the Amount of R for 1 Year; R being the Amount of 1. For the same Reason R^2 forborn another Year will amount to $R \times R \times R$, or R^3; for $1 : R :: R^2 : R^3$, and so on, to any Number of Years, which being called t, the Amount of 1 *l.* for t Years, is R^t: consequently the Amount of any other Principal R for t Years, is $p \times R^t$; for $1 : R^t :: p : p \times R^t$. Or thus: R being the Amount of 1 *l.* for 1 Year, pR is that of p. for $1 : R :: p : pR$. Again; $1 : R :: pR : pR^2$, the Amount of p for 2 Years, and so on; *i.e.* Universally it is pR^t.

SCHOLIUMS.

1. The successive Amounts of any Principal for 1, 2, 3, *&c.* Years, make a Geometrical Progression, whose Ratio is the Amount of 1 *l.* for 1 Year. Thus: The Series of the Amounts of 1 *l.* is $R : R^2 : R^3 : R^4$: *&c.* and of any other Principal p, it is $pR : pR^2 : pR^3 : pR^4$, *&c.* And if we make the Principal the 1st Term, the whole is a Geometrical Progression, *viz.* $1 : R : R^2 : R^3$, *&c.* or $p : pR : pR^2 : pR^3$, *&c.*

2. If the Rate of Interest is determined to any other Time than a Year, as $\frac{1}{2}$, or $\frac{1}{4}$ of a Year, the Rule is the same; and then t represents that stated Time.

But whatever the stated Time is, it remains to be explained, how the Interest or Amount of any Sum is to be calculated for a lesser time; I shall suppose 1 Year the stated Time, because it is so in Law; and whatever the Rules are for the Parts of a Year, they are equally applicable to the Parts of any other Time, to which the Rate of Interest may be supposed to be determined.

Of

Of the Compound Intereſt *or* Amount *of any Sum for the Parts of a Year; the Rate of Intereſt being determined to* 1 *Year.*

Method 1. Some will have it done in Simple Proportion to the Time, or, as ſimple Intereſt; becauſe, ſay they, Compound Intereſt muſt ſuppoſe a certain Time for which Intereſt is at a ſimple ſtated Rate, and after that becomes a Principal bearing Intereſt; ſo that this Time being 1 Year, there can be no ſuch thing as Compound Intereſt for any Part of a Year.

Wherefore, following this Rule, If the Time of a Queſtion is leſs than 1 Year, the Amount is found by the Rule of Simple Intereſt; and if there are whole Years, and part of a Year, then, having found the Amount for the whole Years (which is only Simple Intereſt if there is but 1 Year), take that Amount, and find what it amounts to at Simple Intereſt for the remaining Time leſs than 1 Year.

Method 2. Others proceed upon this Principle, *viz.* That ſince the Rate of Intereſt is determined to 1 Year, this, ſay they, ſuppoſes that all the Improvement that can be made of any Sum by Intereſt in 1 Year, is the ſtated Rate of 5 or 6, *&c. per Cent.* But if Money is Lent for any Time leſs than 1 Year, and Intereſt received for it in ſimple Proportion to the Time, then, by lending out the Whole again, more will be made by it than 5 or 6 *per Cent.* in 1 Year. Therefore they would have the Amount for any Part of a Year calculated ſo, that if this is again conſidered as a Principal bearing Intereſt, it ſhall, after ſo much Time as the former wants of 1 Year, amount, at the ſame Rate, only to the Sum of the Principal and its Simple Intereſt for 1 Year. For *Example:* The Amount for $\frac{1}{4}$ of a Year is ſuch, that being put out to Intereſt for the remaining $\frac{3}{4}$, and ſuppoſed to bear Compound Intereſt from Quarter to Quarter, at the Rate of Simple Intereſt allowed for the 1ſt Quarter, the Amount at the End of the $\frac{3}{4}$ ſhall only be the Sum of the Principal and 1 Year's Simple Intereſt, at the allowed Rate of 5 or 6, &c. *per Cent.*

How this may be done, I ſhall preſently explain; after this general Reflection, *viz.* That as the Law determines the Intereſt of Money lying in the hands of the ſame Perſon to be at Simple Intereſt, ſo any Perſon who puts out Money at Intereſt, and calling it in as oft as poſſible, puts out the Whole again, improves the Money by Compound Intereſt without Breach of Law: Therefore the ſame Reaſon that juſtifies Compound Intereſt from Year to Year, ſeems equally to juſtify it from Quarter to Quarter, or from Day to Day; And ſince the Law allows Intereſt for Parts of a Year in Proportion to the time, it would ſeem alſo by this that the ſtated Intereſt for 1 Quarter or 1 Day, ought to be the proportional Part of one Year's Intereſt, and then the Compound Intereſt to be calculated accordingly from Day to Day. The only Objection to this is, that it is impoſſible for one to lend Money, and be paid from Day to Day, or even by Months or Quarters; and therefore that Foundation is unreaſonable, and conſequently one of the other two Methods muſt be taken; but which of them is to be choſen for the Parts of a Year, let every one determine for themſelves; [for it ſeems to me to depend all upon a Suppoſition] this only I ſhall further obſerve, that as the Simple Intereſt for the Parts of 1 Year is greater than that found by the 2d *Method*, it ſeems reaſonable that he who receives only Simple Intereſt for whole Years, ſhould have the Advantage of the proportional Part for a leſſer Time; and he who has compound Intereſt for whole Years ſhould have Intereſt for the Parts of a Year at a leſſer Rate, by the 2d *Method.*

Now if Intereſt for the Parts of a Year is taken by the 1ſt *Method*, the Rule is already given; but if it's taken upon the other Foundation, then to make the Rule more clear, I ſhall firſt ſuppoſe 1 *l* Principal, and then apply it to other Caſes. Alſo, we muſt diſtinguiſh according as the Time is or is not an aliquot Part of a Year.

Rule 1. If the Time is an aliquot Part of a Year, as $\frac{1}{2}$, $\frac{1}{3}$, or, universally, $\frac{1}{n}$ Part, take the Amount of 1 *l.* for 1 Year, and from it extract the nth Root (*i. e.* the Square Root if it's $\frac{1}{2}$ a Year, the Cube Root if it's $\frac{1}{3}$, the Biquadrate, or 4th Root if it's $\frac{1}{4}$); that Root is the Amount sought.

Example. At 5 *per Cent.* the Amount of 1 *l.* for $\frac{1}{2}$ a Year, is 1.02469, *&c*, this being nearly the Square Root of 1.05

DEMONSTR. $1 : \overline{1.05}^{\frac{1}{2}} :: \overline{1.05}^{\frac{1}{2}} : 1.05$, therefore $\overline{1.05}^{\frac{1}{2}}$ (or 1.02469) put out at Interest for $\frac{1}{2}$ a Year amounts to 1.05, according to the Rate allowed for the 1st Half Year, which is 1.02469, the Principal for the 2d Half Year.

If the Time is $\frac{1}{3}$ of a Year, then, if we suppose two Geometrical Mean Proportionals betwixt 1 and R, thus, $1 : a : b : R$, then must a be the Cube Root of R, from the Nature of Geometrical Progressions; for whatever a is, b is equal to a^2, for $1 : a :: a : a^2$, but it is supposed that $1 : a :: a : b$, therefore $a^2 = b$. Again; R is $= a^3$, for $1 : a :: b : R$, that is $1 : a :: a^2 (= b) : R$, but also $1 : a :: a^2 : a^2 \times a = a^3$, consequently, $R = a^3$, and hence $a = R^{\frac{1}{3}}$. The same Reason is good in all Cases, that is, however many, as n, Terms there be in a Geometrical Progression from 1 to R, as $1 : a : b : c$, &c. : R, or $1 : a : a^2 : a^3$, &c. : R, the first of them, a, is such a Root of R whose Denominator is the Number of Terms, as $R^{\frac{1}{n}}$. Again; According to the supposed Foundation, a, (or $R^{\frac{1}{n}}$) is a Principal, which bearing Compound Interest, at the Rate of $a - 1$ Interest for the nth Part of 1 Year, will amount to R at the Year's End.

2. If the Time is not an aliquot Part of a Year, reduce it to Days, and the 365th Root of R is the Amount for one Day; which Amount raise to that Power whose Index is the Number of Days in the Question, and it is the Amount sought. The Reason is plain from the preceding Case; but the Practice difficult, because of the Difficulty of finding the Root required.

Now for all other principal Sums, their Amount is found from that of 1 *l.* thus; As 1 *l.* to its Amount, so is any other Principal to its Amount; which will be the Product of its Principal, and the Amount of 1 *l.*

Before I leave this Subject, I must observe, That the Extraction required for any aliquot Part of a Year more than $\frac{1}{2}$ or $\frac{1}{3}$ will be tedious by the common Rules, and next to impossible for one Day, which requires the 365th Root. The Algebraic Art furnishes easier Rules for these Extractions; but they go beyond my Limits. The Method of Logarithms will be tolerable exact, which is this, *viz.* Take the Logarithm of the Rate R, divide it by the Denominator of the given aliquot Part of a Year, the Quote is the Logarithm of the Root sought; which therefore is found in the *Table* against that Logarithm.

I shall only add, That if the Time is $\frac{1}{4}$, it may be done at two Extractions of the Square Root, *viz.* Take the Square Root of R, and then the Square Root of the former Root; because the Square Root of the Square Root is the 4th Root, since $2 \times 2 = 4$. Again, from the 4th Root extract the Square Root, it is the 8th Root, and is the Amount for $\frac{1}{8}$ of a Year, and going so on, we may find by the Square Root the Amount for $\frac{1}{16}$, or $\frac{1}{32}$, or $\frac{1}{64}$ *&c.* Part of a Year: But yet these will be tedious above two Extractions. Again, for $\frac{1}{12}$ Part, or one Month, extract the Cube Root of R, then of this the Square Root, and of this again the Square Root, and this last is the 12th Root, or the Amount for one Month, because $3 \times 2 \times 2 = 12$.

And this I presume will be going near enough for *Compound* Interest; so that what Time there is less than one Month, or over any Number of Months, take the Interest of the last Amount for that Time at Simple Interest, the Difference will be inconsiderable in any Case that can occur in common Affairs. But, lastly, I must observe, That the great and

and allowed Use of Compound Intereſt being in the Purchaſe of Annuities, which ſeldom, if ever, go lower than Quarters: It is enough that we have a Rule eaſy enough for that, *viz.* by two Extractions of the Square Root.

PROBL. 2. Having the Amount, Rate, and Time, to find the Principal.

Rule. Divide the given Amount by the Amount of 1 *l.* for the given Time and Rate, *i. e.* by ſuch a Power of the Rate whoſe Index is the Number of Years; the Quote is the Principal ſought.

Univerſally, $\frac{A}{R^t}=p$.

Example. What principal Sum will amount to 201.9963136 *l.* (or 201 *l.* 19 *ſh.* 11 *d.* nearly) in 4 Years, at the Rate of 6 *per Cent.* Compound Intereſt? *Anſwer:* 160 *l.* Thus; The Rate is 1.06, and $\overline{1.06}^4 = 1.26247696$; then $\frac{201.9963136}{1.26247696}=160$.

DEMONSTR. By *Probl.* 1. $A=p\times R^t$, hence, dividing both by R^t it is $\frac{A}{R^t}=p$.

Or thus, by Proportion; $R^t : 1 :: A : \frac{A}{R^t}=p$, becauſe of the Proportion of Principals and their Amounts.

SCHOLIUMS.

1. This *Problem* is the ſame thing as finding the preſent Worth of a Debt due at the End of a certain Number of Years, diſcompting Compound Intereſt: For that preſent Worth muſt be a Sum, which, conſidered as a Principal, will, at the End of the given Number of Years, amount to the Debt; therefore the Difference of the Principal and Amount is the *Diſcompt.*

2. To divide by R^t, is the ſame thing in effect as to divide firſt by R, and this Quote again by R, and ſo on, ſtill dividing by R, till the Number of Diviſions be equal to t; and from this, (or alſo from the Nature of Compound Intereſt, which is again a Proof of this) it follows, that any Debt or Sum due after a certain Number of Years, with the preſent Worths of it for 1, 2, 3, *&c.* Years before it falls due, make a Geometrical Progreſſion decreaſing from the given Sum in the Ratio of R to 1. Thus, in the Series $A : \frac{A}{R} : \frac{A}{R^2} : \frac{A}{R^3}$, *&c.* if A is a Debt payable after 1, or 2, *&c.* Years, then is $\frac{A}{R}$ the preſent Worth of it, diſcompting for 1 Year, $\frac{A}{R^2}$ the preſent Worth, diſcompting for 2 Years, and ſo on; and this is alſo another *Demonſtration* of the *Rule.*

3. If the Time given is leſs than 1 Year, the preſent Worth is to be found either by the Rule of Simple Intereſt; or (if it's ſo agreed upon, or thought more reaſonable) upon the ſame Principle with the other Method of finding the Amount for Time leſs than 1 Year, thus; Find the Amount of 1 *l.* for the given Time; then as that Amount is to 1 *l.* ſo is the given Amount to the preſent Worth ſought; which is plainly the Quote of the given Sum by the Amount of 1 *l.*

PROBL. 3. Having the Principal, Amount, and Rate, to find the Time.

Rule. Divide the Amount by the Principal, then multiply the Rate by itſelf continually till the Product is equal to the preceding Quote; the Index of that Power of the Rate thus produced, or the Number of Multiplications more 1, is the Time.

Example. At 5 *per Cent.* Compound Intereſt, in what Time will 50 *l.* amount to 60 *l.* 15 *ſh.* 6$\frac{1}{40}$ *d.* or 60.7753125 *l.*? *Anſwer:* 4 Years: For $\frac{60.7753125}{50} = 1.21550625 = \overline{1.05}^4$ or $1.05 \times 1.05 \times 1.05 \times 1.05$.

 DEMON.

DEMONSTR. By *Probl.* 1. $A = p \times R^t$; divide both by p, and it is $\frac{A}{p} = R^t$; whence the rest of the Rule is manifest.

SCHOLIUM. If none of the Powers of R is found exactly equal to $\frac{A}{p}$, this shews that the given Principal cannot make the given Amount in any whole Number of Years; but that besides the Number of Years expressed by the Index of that Power of R, which is next less than $\frac{A}{p}$, there must be allowed moreover some Part of a Year; and to find what that is, multiply that Power of R, suppose R^t, by the Principal p, the Product pR^t is the Amount of p for t Years (by *Probl.* 1.); wherefore take the Difference of pR^t, and the given Amount A, *viz.* $A - pR^t$; find in what Time pR^t will amount to A, reckoning by Simple Interest, *i. e.* in what Time it will give $A - pR^t$ of Interest at the given Rate, and that is the additional Time sought. Or if you chuse the other Method, then, to do it to the greatest Exactness, we must know the 365th Root of R; and having found a 4th Proportional to these, *viz.* $pR^t : A :: 1$, which is $\frac{A}{pR^t}$; take the 365th Root of R, and multiply it by it self continually till the Product is equal to $\frac{A}{pR^t}$ or next less, the Number of Multiplications more 1 is the Number of Days sought. But we may more easily do it within 10 or 11 Days by two Extractions of the Square Root, and two of the Cube Roo , thus; Take the Square Root of the Square Root of R; then of this take the Cube Root of the Cube Root, and you have the 36th Root, (for $2 \times 2 \times 3 \times 3 = 36$); multiply this by it self continually till the Product is equal to or next less than $\frac{A}{pR^t}$, the Number of Multiplications more 1, shew nearly how many times 10 Days are contained in the Time sought, because $36 \times 10 = 360$, which is nearly a Year. Or I shall propose another Method: Find the 8th Root of R, (*i. e.* the Square Root of the Square Root of the Square Root); involve it till it be equal to $\frac{A}{pR^t}$, or next less: If you find a Product equal, then the Number of Multiplications, which cannot exceed 7, shew how many times $45\frac{5}{8}$ Days is in the Time sought (for $365 \div 8 = 45\frac{5}{8}$): But if there is none of the Products equal to $\frac{A}{pR^t}$, take the next less, and let us represent it by $R^{\frac{n}{8}}$ (it is the Amount of 1 *l.* for so many times $45\frac{5}{8}$ Days as the Number of Multiplications); take the Difference of this Product and $\frac{A}{pR^t}$, *viz.* $\frac{A}{pR^t} - R^{\frac{n}{8}}$; then find by Simple Interest in what Time $R^{\frac{n}{8}}$ will amount to $\frac{A}{pR^t}$, that is, in what Time it will yield a Sum of Interest equal to $\frac{A}{pR^t} - R^{\frac{n}{8}}$. The *Answer* of this will give a Number of Days, which, added to the Number of Days in as many Times $45\frac{5}{8}$ as the Number of Multiplications that produced $R^{\frac{n}{8}}$ (which are $n - 1$) gives the whole Number of Days nearly; for tho' the last Part of the Work is by simple Interest, yet being within 45 Days it can make no considerable Difference unless the Principal were very large. Or, to add no more, If we find the 12th Root, and use it as above directed for the 8th, we shall find the Time by Compound Interest within $30\frac{5}{12}$ Days, and then find the rest by Simple Interest.

In all these Methods the Logarithms will be very convenient for the Extractions of the Roots of R, and again involving this Root into it self, as the Rules prescribe.

PROBL. 4. Having the Principal, Amount, and Time, to find the Rate.

Rule.

Rule. Take the Quote of the Amount divided by the Principal, and extract a Root of it whose Denominator is the Number of Years; that is the Rate.

Example: At what Rate of Compound Interest will 50 *l.* amount to 60.7753125 *l.* in 4 Years? *Answer: 5 per Cent.* for $\frac{60.7753125}{50} = 1.21550625$, whose 4th Root is 1.05, the Rate or Amount of 1 *l.* for 1 Year.

Demonstr. Since $\frac{A}{p} = R^t$ by the last, and the t Root of R^t is R, therefore the same Root of $\frac{A}{p}$ is R.

Observe, As the first two *Problems* are the most useful, so their Answers are more easily found, and also more determinate.

Scholium. As the Amount of any Principal is the Sum of the Principal and Interest, so if in any of the preceding *Problems* the Interest is sought or given instead of the Amount, the *Answer* is easily found from the preceding; which I shall briefly explain, thus:

Probl. 5. Having the Principal, Rate, and Time, to find the Interest.

Rule. Find the Amount by *Probl.* 1. the Difference of this and the Principal is the Interest.

Probl. 6. Having the Interest, Time, and Rate, to find the Principal.

Rule. Find the Amount of 1 *l.* for the given Time and Rate, then the Difference of 1 *l.* and that Amount being the Interest of 1 *l.* say as that Interest is to 1 *l.* so is the given Interest to its Principal sought.

Probl. 7. Having the Principal, Interest, and Rate, to find the Time.

Rule. The Sum of Principal and Interest is the Amount; by which, with the Rate and Principal, find the Time by *Probl.* 3.

Probl. 8. Having the Principal, Interest, and Time, to find the Rate.

Rule. Find the Amount, as in the last, and then apply *Probl.* 4.

§. 2. Of Annuities.

Defin. 1. An Annuity is a Sum of Money payable every Year for a certain Number of Years, or for ever. And tho' it be divided into half Years or quarters Payments, it still goes under the general Name of Annuity, because the whole Payments make so much in a Year.

I shall first consider the Supposition of yearly Payments, and then other Cases; and also first consider Annuities to continue for a certain determined Number of Years.

2. An Annuity is said to be in *Arrears,* when the Debtor keeps it in his hands for a certain Number of Years, paying the whole at last with Interest for every Year after it falls due; and the Total of the several Years with the Interest due upon each, is called the *Amount* of the Annuity forborn for that time. Again; If an Annuity is to be bought off, or paid all at once, at the very Beginning of the first Year, the Price which ought to be paid for it, discompting for the Advancement, is called the present Worth of it for so many Years.

But then, as either Simple or Compound Intereſt may be made a ſuppoſed Condition in the Queſtion, we muſt accordingly diſtinguiſh; tho' Simple Intereſt, eſpecially in the purchaſing of Annuities, is very unjuſt, as ſhall be fully demonſtrated.

PART. I. *Of* Annuities *at Simple Intereſt.*

1. *Of* Annuities *in Arrears at Simple Intereſt.*

PROB. I. Having the Annuity, Time, and Rate of Intereſt (*i. e.* the intereſt of 1 *l.* for one Year) to find the Amount.

Rule Take the natural Series of Numbers 1, 2, 3, *&c.* to the Number of Years leſs 1. Multiply the Sum of this Series by one Year's Intereſt of the Annuity, (which is the Rate, if the Annuity is 1 *l.*; but it is the Product of the Rate and Annuity, if the Annuity is not 1 *l*) this Product is the whole Intereſt due upon the Annuity. To which add the whole Annuities, (*i e.* the Product of the Annuity and Time) the Sum is the Amount ſought. And *obſerve*, That as the Series 1, 2, 3, *&c.* is an Arithmetical Progreſſion, if you find its Sum by the Rule of *Prob.* 5. *Chap.* 2. *Book* IV. (which is this, multiply the Sum of the Extremes by the Number of Terms, the half of the Product is the Sum) the Work is thereby eaſier.

Example. What is the Amount of 50 *l.* Annuity for 7 Years, allowing Simple Intereſt at 5 *l. per Cent* for every Year after it falls due? Anſwer, 402 *l.* 10 *s.* Found thus: $1+2+3+4+5+6=21$; then $.05\times 50=2.5$, and $2.5\times 21=52.5$. Again, $50\times 7=350$, and $350+52.5=402.5$, or 402 *l.* 10 *s.* the Amount ſought.

DEMONSTR. Whatever the Time is, there is due upon the firſt Year's Annuity as many Years Intereſt as the whole Number of Years leſs 1; and gradually 1 leſs upon every ſucceeding Year, to the laſt but one; upon which there is due one Year's Intereſt, and none upon the laſt; wherefore in whole there is due as many Years Intereſt of the Annuity, as the Sum of the Series 1, 2, 3, *&c.* to the Number of Years leſs 1. Conſequently one Year's Intereſt multiplied by this Sum muſt be the whole Intereſt due. To which the whole Annuities added, the Sum is plainly the Amount.

SCHOL. This *Problem* may be ſolved alſo in this manner, *viz.* Take an Year's Intereſt of the Annuity for the leaſt Term, and alſo the common Difference of an Arithmetical Progreſſion carried to as many Terms as the Number of Years leſs 1, (*i.e.* take one Year's Intereſt of the Annuity, then double it; then take it three times, and ſo on, till you take it as oft as the Number of Years leſs 1) its Sum is the whole Intereſt due, as is plain by what's already ſhewn; or appears by this, that it is the ſame thing to multiply by the Sum of the Series 1, 2, 3, *&c.* as in the 1ſt *Method*, or by each of the Terms ſeparately; and then add the Products, which is the other *Method*. Again; It will come to the ſame thing, if we take an Arithmetical Progreſſion, whoſe leaſt Term, and alſo the common Difference, is the Rate, and carry it to the ſame Number of Terms as before; the Sum of this Series is the whole Intereſt due upon 1 *l.* Annuity, (as appears by the laſt *Method*); wherefore if we multiply this Sum by any other given Annuity, the Product is the whole Intereſt due upon this other Annuity. Or take the Reaſon of it thus: Since one Year's Intereſt of the Annuity is the Product of the Rate and Annuity, (as is plain by the common Rules) it is the ſame thing to multiply each Term of the Series 1, 2, 3, *&c.* by that Product; and then add all the Products, which is the 2d *Method*; or to multiply them firſt by the Rate, and then each of theſe Products, or their Sum, by the Annuity; which is the laſt *Method.*

2. But I ſhall again more briefly ſhew the Coincidence of all theſe three *Methods* by the *univerſal Method* of Expreſſion, (which is neceſſary for the ſake of the following *Problems*) and alſo bring them all into one general Rule. Thus:

Let

Let r be the Rate, n the Annuity, t the Time, and A the Amount; then is rn one Year's Interest of the Annuity, (for $1:r::n:rn$) and so the 1st Rule may be expressed thus: $\overline{1+2+3, \&c. +\overline{t-1}}\times rn+tn=A$. Again; The 2d Rule is $rn+2rn+3rn, \&c. +\overline{t-1}\times rn+tn=A$, which coincides with the first, because $rn+2rn+3rn, \&c. +\overline{t-1}\times rn = \overline{1+2+3, \&c. +t-1}\times rn$.) Lastly, the 3d Rule is, $\overline{r+2r+3r, \&c. +\overline{t-1}\times r}\times n+tn=A$, (which coincides with the 2d, because $rn+2rn+3rn, \&c. +\overline{t-1}\times rn = \overline{r+2r+3r, \&c. +\overline{t-1}\times r}\times n$.)

In the last place, to bring all these into one general Rule. 1. The Sum of the Series $1+2+3, \&c. +\overline{t-1}$, is by the Rules of Progression $\overline{t-1}+1\ (=t)\times\frac{t-1}{2}=\frac{t^2-t}{2}$; and this multiplied by rn, is $\frac{t^2rn-trn}{2}$ the Sum of the Interest due. 2. In the 2d Rule, the Sum of the Series $rn+2rn+3rn, \&c. +\overline{t-1}\times rn$ is $\overline{t-1}\times rn+rn\ (=trn)\times\frac{t-1}{2}=\frac{t^2rn-trn}{2}$ the Sum of the Interest as before. 3. In the 3d Rule, the Sum of the Series $r+2r+3r, \&c. +\overline{t-1}\times r$ is $\overline{t-1}\times r+r\ (=tr)\times\frac{\overline{t-1}}{2}=\frac{t^2r-tr}{2}$; which multiplied by n produces $\frac{t^2rn-trn}{2}$ the whole Interest as before. If now to this Sum of the Interest we add the whole Annuities, the Sum is $\frac{t^2rn-trn}{2}+tn=A$; which is the universal Rule. In Words thus:

Multiply the Time, Rate and Annuity continually. Then multiply this Product by the Time; subtract the first Product from the 2d, and take the half of the Difference; to which add the Product of the Time and Annuity, the Sum is the Amount.

PROB 2. Having the Amount, Rate and Time, to find the Annuity.

Rule. Take any Annuity at pleasure, and, by *Prob.* 1. find its Amount; then, by the Rule of 3, as this Amount is to its Annuity, so is the given Amount to its Annuity. So that if you take 1 *l.* Annuity, by its Amount divide the given Amount, the Quote is the Annuity sought.

Example. What Annuity will in 7 Years amount to 402 *l.* 10 *s.* allowing 5 *per Cent* Simple Interest? Answer, 50 *l.* Found thus: The Amount of 1 *l.* Annuity is 8.05 *l.* for by *Prob.* 1. $1+2+3+4+5+6=21$, and $21\times .05=1.05$; then $1.05+7=8.05$, the Amount of 1 *l.* Annuity. Lastly, As $8.05:1::402.5:\frac{402.5}{8.05}=50$ the Annuity sought.

DEMONSTR. The Reason of this Rule is plain, since to every Part of an Annuity there must necessarily correspond a proportional Part of the Amount.

SCHOLIUM. The *Problem* may be also solved thus: Take the Sum of the Series 1, 2, 3, *&c.* to the Number of Years less 1; this Sum multiply by the Rate, and to the Product add the Years; and by this Sum divide the Amount, the Quote is the Annuity.

For, by *Problem* 1. $A=\overline{1+2+3, \&c. +\overline{t-1}}\times rn+tn$; which is $=\overline{\overline{1+2+3, \&c. \overline{t-1}}\times r+t}\times n$; and dividing both by $\overline{1+2+3, \&c. +\overline{t-1}}\times r+t$, it is $\frac{A}{\overline{1+2+3, \&c. \overline{t-1}}\times r+t}=n$.

Again;

Again; Because $1+2+3$, &c. $+\overline{t-1}=\frac{t^2-t}{2}$; this multiplied by r is $\frac{t^2r-tr}{2}$; to which add t, the Sum is $\frac{t^2r-tr+2t}{2}$. By which dividing A the Quote is $\frac{2A}{t^2r-tr+2t}=n$, which is another *Rule*; and this may also be immediately deduced from the general Rule of *Prob.* 1. *viz.* from this Equation $A=\frac{t^2rn-trn}{2}+tn$, or $\frac{t^2rn-trn+2tn}{2}$. For multiplying both by 2, it is $2A=t^2rn-trn+2tn$; and dividing by $t^2r-tr+2t$, it is $\frac{2A}{t^2r-tr+2t}=n$.

PROB. 3. Having the Annuity, Amount, and Time, to find the Rate.

Rule. Take the Difference betwixt the Amount, and the Product of the Annuity and Time; the Difference divide by the Product of the Annuity multiplied into the Sum of this Series 1, 2, 3, *&c.* to the Number of Years less 1, the Quote is the Rate.

Example. At what Rate of Interest will an Annuity of 50 *l.* amount to 402 *l.* 10*s.* in 7 Years? Answer, 5 *per Cent.* or .05 to 1 *l.* Thus, $50\times 7=350$, then $402.5-350=52.5$. Again, $1+2+3+4+5+6=21$, and $21\times 50=1050$. Then, lastly, $\frac{52.5}{1050}=.05$.

DEMON. By *Prob.* 1. $A=\underline{1+2+3,\ \&c.\ +\overline{t-1}}\times rn+tn$. Take tn from both, and it is $A-tn=\underline{1+2+3,\ \&c.\ +\overline{t-1}}\times rn$. Divide both by $\underline{1+2+3,\ \&c.\ +\overline{t-1}}\times n$, and it is $\frac{A-tn}{\underline{1+2+3,\ \&c.\ +\overline{t-1}}\times n}=r$, which is the Rule.

SCHOLIUM. $1+2+3$, *&c.* $+\overline{t-1}=\frac{t^2-t}{2}$; and this multiplied by n is $\frac{t^2n-tn}{2}$. By which divide $A-tn$, the Quote is $\frac{2A-2tn}{t^2n-tn}=r$, which is another *Rule*; and this may be deduced from the general Rule of *Prob.* 1. thus, $A=\frac{t^2rn-trn}{2}+tn$. Hence $A-tn=\frac{t^2rn-trn}{2}$, and $2A-2tn=t^2rn-trn$; and lastly, $\frac{2A-2tn}{t^2n-tn}=r$.

PROB. 4. Having the Annuity, Amount and Rate, to find the Time.

Rule. Take the Product of the Annuity and Rate, as the first Term of a Progression, whereof the same Product is the common Difference; at every Step from the very first Term, take the Sum of the Series, and to it add the Product of the Annuity multiplied into a Number more by 1 than the Number of Terms summed. Go on in this manner till you find a Sum equal to the given Amount, and the Number multiplied into the Annuity in that last Step is the time sought.

Example. In what time will 50 *l.* Annuity amount to 402 *l.* 10*s.* at 5 *per Cent.*? Answer, 7 Years. Found thus:

$50\times .05=2.5$: then for the rest of the Operation, see it in the Margin.

	rn	$2rn$	$3rn$	$4rn$	$5rn$	$6rn$
	2.5 +	5 +	7.5 +	10 +	12.5 +	15.
Sums,		7.5	15	25	37.5	52.5.
Products of 50,		150	200	250	300	350 (=50×7.)
2d Sums,		157.5	215	275	337.5	402.5.

DEMONSTR. By *Prob.* 1. we have $A=rn+2rn+3rn$, &c. $+\overline{t-1}\times rn+tn$; and if the Time is but 1 Year, then is $A=n$. If it is 2 Years, then is $A=rn+2n$; if 3 Years, then is $A=rn+2rn+3n$, and so on; which makes the Rule manifest.

SCHO-

SCHOLIUMS.

1. If you never find a Sum equal to the Amount, then the *Problem* is impossible in whole Years.

2. The preceding *Problem* will be tedious if there are many Years.

There is another Rule easier in Practice, tho' not so simple in its Demonstration. It is deduced from *Problem* 1. thus, $A = \frac{rnt^2 + 2tn - trn}{2}$, whence $2A = rnt^2 + 2tn - trn = rnt^2 + \overline{2n - rn} \times t$; divide by rn, and it is $\frac{2A}{rn} = t^2 + \frac{2n - rn}{rn} \times t$: Call $\frac{2n - rn}{rn} = d$, then it is $\frac{2A}{rn} = t^2 + dt$, and by *Probl.* 6. *Ch.* 2. *Book* III. $t = \overline{\frac{2A}{rn} + \frac{dd}{4}}\Big|^{\frac{1}{2}} - \frac{d}{2}$.

2. *Of the Purchase of* Annuities *at Simple Interest.*

In the following *Problems*, the Things concerned are the Annuity, it's present Worth, the Time of Continuance, and Rate of Interest allowed to the Purchaser for the Advancement of his Money.

As before, we shall express the Annuity by n, the Time t, Rate r, and the present Worth p.

PROBL. 5. Having the Annuity, Rate, and Time, to find the present Worth.

For the Solution of this *Problem*, there are two different Rules given by different Authors; but as they give different Answers, they cannot be both right. I shall first lay down the *Rules*, and then examine which is the right one.

Rule 1. Find the present Worth of each Year by it self, discompting from the Time it falls due (by *Probl.* 8. *Part* 1. §. 1.); the Sum of all these is the present Worth sought.

Example: What is the present Worth of an Annuity of 100 *l.* to continue 5 Years, discompting at the Rate of 6 *per Cent.* or .06 to 1 *l.*? *Answer:* 425 *l.* 18 *s.* 9 *d.* 2 *f.* nearest; in Decimals 425.93932, *&c.* Found thus:

		as		to		so is		to	
The Amount of 1 *l.* for	1 Year,	1.06	:	1	::	100	:	94.33962	
	2 Years,	1.12	:	1	::	100	:	89.28571	
	3 Years,	1.18	:	1	::	100	:	84.74576	&c.
	4 Years,	1.24	:	1	::	100	:	80.64516	
	5 Years,	1. 3	:	1	::	100	:	76.92307	
								425.93932	

Observe, The Work will be a little easier, if you find the present Worth of 1 *l.* Annuity for the given Time, and then multiply that by the given Annuity, the Product is the present Worth of it, because of the Proportionality of Annuities and their present Worths.

Rule 2. Find what each Year's Annuity would amount to, being forborn to the End of the last Year, allowing Simple Interest from the time it falls due; *that is,* find the Amount of the Annuity as forborn the whole Time (by *Probl.* 1. of this §.) then find the present Worth of that Amount as a Sum due at the End of the given Time.

In

In the preceding *Example* the Amount of 100*l.* Annuity for 5 Years at 6 *per Cent.* is 560*l.* *viz.* the 5 Years Annuities 500*l.* and 60*l.* which is 10 Years Intereſt of 100*l.* Then the Amount of 1*l.* forborn 5 Years is 1.3; and as 1.3 is to 1, ſo is 560 to 430.76923, *&c.* or 430*l.* 15*ſh.* 4*d.* 2*f.* neareſt the preſent Worth by this Rule.

SCHOLIUM. The firſt *Rule* is Mr. *Kerſey*'s in his *Appendix to* Wingate's *Arithmetick*: The Second has for its Author or Defender Sir *Samuel Moreland*, in his *Doctrine of Intereſt*, who wonders how ſuch groſs Miſtakes, as he calls *Kerſey*'s Rule, could paſs thro' the Hands of ſo many learned and ingenious Artiſts. After him, others have taken the ſame Method, and particularly Mr. *Ward*, who makes this Remark, *That Sir* Samuel Moreland *has detected ſeveral material Errors in* Kerſey *and others in the Buſineſs of Intereſt*: But to my Apprehenſion the Error lies all upon Sir *Samuel*'s Side.

Before I enter upon the Examination of the Reaſons of theſe Rules, I muſt firſt *obſerve*, That by ſaying, *Kerſey*'s Rule gives the true Price of the Annuity, I mean only, That it is true in conſiſtence with the ſuppoſed Condition or Agreement of allowing Simple Intereſt, and not abſolutely ſo: For if we enquire what, in ſtrict Equity and Juſtice, ought to be paid for the Annuity, then neither of theſe Rules ſhew it; for they both give too much, and the true Price muſt be found by diſcompting Compound Intereſt (as *Moreland* acknowledges), and then both the Methods give the ſame Anſwer. Nor is this contrary to Law: For tho' when an Annuity is in Arrears the Law forbids taking Compound Intereſt, yet in the Purchaſe of an Annuity, if the Buyer offers ſuch a Price as allows him Compound Intereſt for the Advancement of his Money, he does nothing contrary to Law; becauſe in Buying one may offer what Price he thinks fit: And he has this good Reaſon for it, that by putting out his Money, and *lifting* it at every Year's End, he can improve it by Compound Intereſt.

But to ſhew further how unjuſt Simple Intereſt is in the Purchaſe of Annuities, take this *Example*: An Annuity of 50*l.* is to be bought for 40 Years, diſcompting Simple Intereſt at 5 *per Cent*: The Price according to *Moreland*'s Rule is 1316*l.* 13*ſh.* 4*d.* a Sum of which one Year's Intereſt exceeds the Annuity: Would not then one think he had made a pretty Bargain, to give for an Annuity to continue only 40 Years, a Sum which would yield him a greater Yearly Intereſt for ever? If it's alſo calculated by the other Rule, the ſame will happen, as I have actually found; tho' it is much leſs than the other, for it does not exceed 1100*l.*

This *Example* may, I think, be ſufficient to ſhew the Abſurdity of diſcompting for Annuities at Simple Intereſt, and conſequently to put that Practice quite out. And this too might perhaps be a good enough Reaſon for not troubling you any further with the Compariſon of the preceding *Rules*, and ſo to paſs on to Compound Intereſt. But as it is a Queſtion that belongs to Arithmetick, to find the Price of an Annuity upon the Suppoſition of Simple Intereſt, which is to be found in every Book upon this Subject, and ſince theſe two Rules are become Matter of Diſpute among Authors, I thought I could not reaſonably omit the Examination of theſe, and ſhewing why I prefer *Kerſey*'s.

The Reaſons *of the preceding* Rules *examined.*

Mr. *Kerſey* takes the Reaſon of his Rule for a thing of it ſelf manifeſt from the Nature and Rules of Diſcompt; for if that is right, he conſiders every Year by it ſelf, as ſo many ſingle and independent Debts, due after 1, 2, 3, *&c.* Years, ſo that the preſent Worth of each being found, the Sum of theſe muſt be the preſent Worth of the Whole; which ſeems to be the plain ſtate of the Queſtion.

Sir *Samuel Moreland* ſays this Rule is groſly wrong, becauſe there is no Conſideration had of the *Forbearance* of Intereſt, (*i. e.* of the Annuity); and therefore he propoſes the conſidering what the Annuity will amount to, being forborn for the whole Time of the Queſtion; and then he ſuppoſes that all muſt agree, That whatever be the preſent Worth of the Annuity, it muſt be ſuch a Sum, as, put out to Intereſt for the Number of Years

the Annuity continues, will amount to the ſame Sum the Annuity does; becauſe the Seller ought to have a Price by which he can make as much at Simple Intereſt during the Continuance of the Annuity, as he could have made by *lifting* every Year of his Annuity as it falls due, and putting it out to Intereſt for the remaining Years of the Queſtion.

This is the firſt and chief Argument which Sir *Samuel* uſes; he adds ſomething further concerning the preſent Worth of the ſeveral Years, which I ſhall conſider afterwards: But as to this firſt Argument, I am ſo far from agreeing to his Suppoſition, that I think the very contrary of what he objects to *Kerſey*'s Rule is a juſt Objection to his own; becauſe I think the Conſideration of the Forbearance of the Annuity is a thing altogether out of the Queſtion: For in purchaſing an Annuity, does the Purchaſer any other thing than buy ſeveral Sums of Money to be paid to him at ſeveral Times, for which he advances ready Money all at once? And this being plainly the Caſe, what has he to do with the Conſideration of the Forbearance of the Annuity? Does he not fairly pay for the Whole when he pays for each Sum ſeparately, diſcompting from the Time it falls due, or payable to him? For thus he pays for each Year ſuch a Sum as will amount to that Year when it falls due: And each Year's Price being conſidered as a Principal Sum bearing Intereſt in the Seller's Hands, will make him as much Debtor to the Buyer at each Year's End, as the Annuity then due and received by the Buyer makes him in the Seller's Debt; and conſequently the Buyer is never in Arrears with the Seller, and ſo has no Buſineſs with the Conſideration of Forbearance or Amount of the Annuity: The Seller being thus cleared at every Year's End, let him make the beſt he can of it.

Again; The Argument from the Amount of the Annuity would be juſt as good if you extend it to 7 or any Number of Years after the laſt Year in the Queſtion: For the Whole being in Arrears 7 Years longer, would have a greater Amount; and this conſidered as a Debt payable after ſo many Years, will have a greater preſent Worth: And ſince the Annuity in the Queſtion will make ſuch an Amount, being forborn ſo long after the Time of the Annuity, why ſhould not the Seller inſiſt upon a Price that will Amount to the ſame Sum in the ſame Time? But if this would be ridiculous, the other is equally ſo; for I have no more to do with the Conſideration of any Year being in Arrears after it is payable (and indeed actually paid) to the End of the laſt Year, than with that Year, or all the Years being forborn after the laſt Year in the Queſtion.

I ſhall in the next place ſhew the Fallacy of *Moreland*'s Argument from his own Conceſſions. He owns that the Preſent Worth of a ſingle 100 *l.* due at one Year's End, of another ſingle 100 *l.* due after 2 Years, and another 100 *l.* due after 3 Years, and ſo on, are all juſtly found according to *Kerſey*'s Rule: And if ſo, pray where is the Difference betwixt an Annuity of 100 *l.* and ſo many ſingle 100 *l.*'s due after 1, 2, 3, *&c.* Years? It is beyond Queſtion, the Caſes are the ſame if theſe ſeveral Debts are all owing by one Perſon. But perhaps it will be ſaid, that the Conceſſion is made only upon Suppoſition that theſe ſeveral 100 *l.*'s are due by different Perſons; which ſeems to be *Moreland*'s Meaning by calling it a ſingle 100 *l.* (*i. e.* as I now take it, 100 *l.* in one Man's Hands, and another 100 *l.* in another Man's Hands, *&c.*) Now to ſhew that this can make no Alteration of the Caſe: Suppoſe 5 Men owe you each 100 *l.* payable one at 1 Year's End, another at 2 Years End, *&c.* the Preſent Worths to be paid by the ſeveral Debtors for their ſeveral Debts, are, it is own'd, according to *Kerſey*'s Rule: Suppoſe next, that I would buy the Right of theſe Debts, paying them *per Advance*, can there be any more juſtly asked of me than of the ſeveral Debtors? And is there any manner of Difference, either as to the Buyer or Seller, betwixt this Caſe, and an Annuity of 100 *l.*? For in either Caſe I buy 100 *l.* to be paid me after 1 Year, another 100 *l.* to be paid after 2 Years, and ſo on. And on the other hand, the Seller has the ſame Argument with me from the Amount of theſe Debts forborn till the Time the laſt is payable, as from the Amount of an Annuity payable by one Debtor; and yet it is deſtroyed in this Caſe, by acknowledging that the ſeveral Debtors ought to pay by *Kerſey*'s Rule. Again; Take this other *Example:* Suppoſe

I owe to each of 5 different Persons 100 *l.* payable at 1, 2, 3, 4, 5 Years End; if I were to pay off these Debts *per Advance*, the Present Worths are acknowledged to be according to *Kersey*'s Rule: Now if one Person gets a Right to all these Debts, it is plain I become Debtor to him in an Annuity of 100 *l.* to continue 5 Years; and if I propose to pay them off at once, by what Reason ought I to pay him more than would have justly satisfied his Constituents?

It remains that I consider *Moreland*'s Argument from the Present Worths of the several Years, from whence he thought the Demonstration perfected; tho' his way of calculating these Present Worths, affords, I think, another strong Argument against the Truth of his Rule.

He supposes an Annuity of 100 *l.* The Present Worth of the 1st Year he allows to be according to *Kersey*'s Rule: For the 2d Year he takes the Amount of the first 2 Years, *viz.* 206 *l.* (supposing 6 *per Cent.*) then finds the Present Worth of 206 *l.* due after 2 Years, which he considers as the Present Worth of the first 2 Years together; from that he takes the Present Worth of the first Year, and calls the Difference the Present Worth of the 2d Year; because the present Worth of the 1st Year taken from the Present Worth of the first two Years, must leave the Present Worth of the 2d Year alone. Then for the 3d Year, he takes the Amount of the first 3 Years, *viz.* 318 *l.* and finding its Present Worth as a Debt due after 3 Years, from this he takes the Present Worth of the first 2 Years, (or of 206 *l.* due after 2 Years) and the Difference he calls the Present Worth of the 3d Year alone; and in this Manner he goes on; then concludes, that because what he calls the Present Worths of the several Years make up his total present Worth, therefore his general Rule is right.

For *Answer* to this, I do acknowledge that the Present Worth of all the Years together, is exactly equal to the Sum of the Present Worths of the several Years considered separately: But then to shew that his Present Worth of the Whole is right, he ought first to have proved, that his Present Worths of the several Years are right; which I deny, and shall prove to be false.

In order to which, *Observe*, That by these Present Worths his Argument requires that he mean the Prices the several Purchasers of the several Years ought to pay for them: For you give the Buyer of the Whole no Reason why he ought to pay the Sum of all these Present Worths, unless you shew him that these are the Prices to be paid by several Purchasers of the several Years; and as he must be satisfied that he ought to pay no less, so there is no manner of Reason why he should pay more. Let us then examine the Justice of these several Present Worths.

The first Year is the same as by *Kersey*'s Rule. But he who buys the 2d Year pays the Difference betwixt the Price of the first Year and the Present Worth of 206 *l.* due after 2 Years; and this 6 *l.* is one Year's Interest of the 1st Year's Annuity, considered as forborn 1 Year; so that the Purchaser of the 2d Year pays most unreasonably for the Consideration of the Forbearance of the 1st Year, with which he has no Concern; and the further from the 1st Year, these Present Worths are the further wrong. It had been more agreeable to his fundamental Supposition of Forbearance, to make the Price of every Year the Present Worth of its Amount, considered as forborn to the End of the whole Years; for the Sum of these is also equal to his Present Worth of the Whole: But this seemed too gross when applied to single Purchasers of the several Years; And is it not rather more absurd to make them pay for the Forbearance of what they have not purchased?

Tho' this Argument is tedious enough already, yet I must take notice of one Case wherein *Moreland*'s Rule would take place; that is, If we suppose the Debtor of an Annuity is obliged to keep it in his Hands, paying Interest to the End of the whole Time; then if he would pay it off, the Creditor has Reason to insist on the Consideration of the Amount of the Annuity as a Debt payable at the End of the Years of the Annuity; but if

if the Annuity is payable as it falls due, he cannot justly insist on that Consideration; and in this Case the Price ought to be by *Kersey's* Rule. So that where the Condition of paying the Annuity is the easiest (as it is certainly more advantageous for the Debtor to keep it in Arrears at Simple Interest, than pay it every Year), the Purchase of it is the dearer; And where the Condition of paying it is the hardest, (*viz.* paying it as it falls due) the Purchase is the easiest; which is something absurd and contradictory: But then this proves neither of the Rules to be wrong; and it arises from the Injustice of Simple Interest; for if Compound Interest is allowed, then to pay the Annuity as it falls due, or keep it in Arrears at Compound Interest, are the same thing; and so also the Present Worths by both Methods are the same. Again; If we suppose either the Creditor or Debtor has it in his Choice, that the Annuity be paid as it falls due, or kept in Arrears at Simple Interest, then the Advantage is upon the same Side both in Paying and Buying: For as the Creditor would certainly chuse to have it paid Yearly, so in selling it he would have Reason to insist on the other side of the Choice, and demand a Price agreeable to that Obligation on the Debtor to keep it in Arrears: Again, As the Debtor would certainly chuse to keep it in Arrears, so in buying it off he would reasonably insist on the other side of the Choice, *viz.* paying it Yearly, because the Creditor cannot oblige him to keep it in Arrears: Yet with Compound Interest there would be no Advantage in having the Choice.

Observe, in the last place, That if another than the Debtor of an Annuity proposes to buy it, then, if the Debtor can keep it in Arrears at Simple Interest, (or is obliged to do so) as this is a real Disadvantage to the Purchaser, so the Seller cannot reasonably insist on that Consideration with him in calculating the Price of it, as he might do if the Debtor buys it; which is another Contradiction, *viz.* that the Price should be different to different Buyers. But this also arises from simple Interest.

SCHOLIUM 2. Tho' I think it's made evident that *Kersey's* Rule is the right one, yet I shall give the Solution of the following three *Problems* upon both Suppositions; and in order to it, I shall here give you the universal Expressions of both these Rules.

The 1st *Rule* is $p = \frac{n}{1+r} + \frac{n}{1+2r} + \frac{n}{1+3r}$, *&c.* to $\frac{n}{1+tr}$.

The *Reason* is this: r being 1 Year's Interest of 1 *l*, $1+r$ is the Amount of 1 *l.* for 1 Year; and therefore the Present Worth of the 1st Year's Annuity, n, is $\frac{n}{1+r}$; for $1+r : 1 :: n : \frac{n}{1+r}$. For the like Reason $\frac{n}{1+2r}$ is the Present Worth of the 2d Year; for $1+2r$ is the Amount of 1 *l.* for 2 Years; and so of the rest; which are the Present Worths of the several Years of the Annuity.

The 2d Rule is $p = \frac{t^2rn + 2tn - trn}{2+2tr}$.

The *Reason* is this: By *Probl.* 1. the Amount of the Annuity is $\frac{t^2rn - trn}{2} + tn$ equal to $\frac{t^2rn - trn + 2tn}{2}$: Then the Present Worth of this, as a Debt payable after t Years, is a 4th Proportional to $1+tr$ (the Amount of 1 *l.* for t Years) 1 *l.* and the Amount of the Annuity, which is according to the *Rule*: For $1+tr : 1 :: \frac{t^2rn + 2tn - trn}{2} : \frac{t^2rn + 2tn - trn}{2+2tr}$.

PROBL. 6. Having the Present Worth, Rate, and Time, to find the Annuity.

Rule. Take any Annuity at pleasure, and find its Present Worth by either of the preceding Rules you please; then, as that present Worth is to its Annuity, so is the given Pre-

sent

ſent Worth to its Annuity: So that if you take 1 *l.* Annuity, then the given Preſent Worth divided by the Preſent Worth of 1 *l.* Annuity, quotes the Annuity ſought.

Example: What Annuity, to continue 5 Years, is worth 220 *l* Preſent Worth, allowing Simple Intereſt at 5 *per Cent.*? *Anſwer:* 50 *l.* 8 *ſh.* nearly, by *Kerſey*'s Rule, and by *Moreland*'s Rule it is preciſely 50 *l.*

Thus, the Preſent Worth of 1 *l.* Annuity for 5 Years, at 5 *per Cent.* is, by *Kerſey*'s Rule, 4.3641 *l.* nearly; then, as 4.3641 to 1, ſo is 220 to 50.399 *l.* nearly, which is nearly 50 *l.* 8 *ſh.*

The Preſent Worth of 1 *l.* Annuity by *Moreland*'s Rule is 4.4 *l*; and as 4.4 : 1 :: 220 : 50 preciſely.

The *Reaſon* of this Rule is manifeſt, becauſe all Annuities and their Preſent Worths are proportional.

PROB. 7. Having the Annuity, preſent Worth, and Rate, to find the Time.

Rule 1. If the preſent Worth is according to *Kerſey*'s Rule, then take the Series of the Amounts of 1 *l.* Principal for 1, 2, 3, *&c.* Years. Divide the given Annuity by each Term of that Series ſucceſſively; and at every Step take the Sum of all theſe Quotes, (*i. e.* add the 1ſt Quote to the 2d, and this Sum to the 3d, and ſo on) going on thus till you find a Sum equal to the given preſent Worth; and the Number of Quotes added is the Number of Years ſought. The Reaſon of which is manifeſt from *Prob.* 5. *Schol.* 2.

Example. What time muſt an Annuity of 50 *l.* 8 *s.* continue, to be worth 220 *l.* ready Money at the Rate of 5 *per Cent.*? Anſwer, 5 Years.

For dividing 50 *l.* 8 *s.* or 50.4 *l.* ſucceſſively by theſe 5 Diviſors, 1.05, 1.1, 1.15, 1.2, 1.25, (the Amounts of 1 *l.* for 1, 2, 3, 4, 5 Years) the Sum of the Quotes make 219 *l.* 19 *s.* 3 *d.* nearly; which is near to 220 *l.* The Reaſon why it makes not preciſely 220 *l.* is that 50 *l.* 8 *s.* is not preciſely the Annuity which has 220 *l.* for its preſent Worth for 5 Years, as we ſaw in the laſt *Problem*: and beſides, the Quotes are here determined only to a certain Degree; but as three of them are compleat, ſo the other two don't want 1 Farthing each.

Rule 2. If the preſent Worth is according to *Moreland*'s Rule, then the Time is to be found thus: Divide 2 by the Ratio; alſo divide the Sum of the Annuity, and twice the preſent Worth by the Annuity. Then take the Difference betwixt theſe two Quotes; and next, to the 4th Part of the Square of that Difference, add the Quote of twice the preſent Worth divided by the Product of the Rate and Annuity. Of this Sum extract the Square Root: From which Root take half of the firſt mention'd Difference, in caſe the Quote of 2 divided by the Ratio be greater than the other Quote; but if it's leſſer, add the half of that Difference to the Root; this laſt Difference or Sum is the Number of Years.

Example. What Time muſt an Annuity of 50 *l.* continue to be worth 220 *l.* at 5 *per Cent.*? Anſwer, 5 Years. Thus:

$2 \div .05 = 40$; then $2 \times 220 = 440$, $50 + 440 = 490$. $490 \div 50 = 9.8$; then $40 - 9.8 = 30.2$, and the Square of $30.2 = 912.04$, of which $\frac{1}{4}$ is 228.01. Next $.05 \times 50 = 2.5$, and $440 \div 2.5 = 176$; then $228.01 + 176 = 404.01$, whoſe Square Root is $= 20.1$; then $30.2 \div 2 = 15.1$; and laſtly, $20.1 - 15.1 = 5$, the Years ſought.

DEMONSTR. By *Problem* 5. (ſee *Schol.* 2. *Rule* 2.) it is $p = \frac{t^2rn + 2tn - trn}{2 + 2tr}$. Whence $2p + 2trp = t^2rn + 2tn - trn$; and again, $2p = t^2rn + 2tn - trn - 2trp$; and, dividing by rn, 'tis $\frac{2p}{rn} = t^2 + \frac{2tn - 2trp - trn}{rn}$, or $t^2 + \frac{2n - 2rp - rn}{rn} \times t$.

But

But $\frac{2n-2rp-rn}{rn}=\frac{2n}{rn}\left(=\frac{2}{r}\right)-\frac{2rp}{rn}\left(=\frac{2p}{n}\right)-\frac{rn}{rn}(=1)$; therefore $\frac{2p}{rn}=$ $t^2+\overline{\frac{2}{r}-\frac{2p}{n}-1}\times t=t^2+\overline{\frac{2}{r}-\frac{2p+n}{n}}\times t$; and if for brevity we call $\frac{2}{r}-\frac{2p+n}{n}$ $=d$; then is $\frac{2p}{rn}=t^2+dt$, and $t=\overline{\frac{2p}{rn}+\frac{dd}{4}}\Big|^{\frac{1}{2}}-\frac{d}{2}$, (by *Prob.* 6. *Chap.* 2. *Book* III.) But if $\frac{2}{r}$ is less than $\frac{2p+n}{n}$, let also $\frac{2p+n}{n}-\frac{2}{r}$ be called d, and then is $\frac{2p}{rn}=t^2-dt$, and $t=\overline{\frac{2p}{rn}+\frac{dd}{4}}\Big|^{\frac{1}{2}}+\frac{d}{2}$, by the same *Problem.*

PROB. 8. Having the Annuity, present Worth and Time, to find the Rate.

1. If the present Worth is taken according to *Kersey*'s Rule, there is no Method within my Limits that will solve the *Problem*, except for that one Case where the Time is two Years. The Rule for which may be easily deduced from *Prob.* 5. *viz.* from this, $p=\frac{n}{1+r}+\frac{n}{1+2r}$. Thus, Adding these two Quotes or Fractions, the Sum is $p=\frac{2n+3rn}{1+3r+2r^2}$. Hence $p+3rp+2pr^2=2n+3rn$, and $2pr^2+3rp=2n+3rn-p$, and $2pr^2+3rp-3rn=2n-p$, and $r^2+\frac{3p-3n}{2p}\times r=\frac{2n-p}{2p}$; and calling $\frac{3p-3n}{2p}$ $=d$, it is $r^2+dr=\frac{2n-p}{2p}$. Whence (by *Problem* 6. *Chapter* 2. *Book* III.) $r=\overline{\frac{2n-p}{2p}+\frac{dd}{4}}\Big|^{\frac{1}{2}}-\frac{d}{2}$.

2. If the present Worth is according to *Moreland*'s Rule, then the Solution of thi *Problem* is deduced from his Rule for *Prob.* 5. *viz.* $p=\frac{nrt^2+2tn-trn}{2+2tr}$. Whence $2p+2ptr=nrt^2+2tn-trn$, and $2ptr-nrt^2+trn=2tn-2p$, and $r=\frac{2tn-2p}{2pt+tn-nt^2}$; that is, take double of the Difference betwixt the present Worth and the Product of Time and Annuity; divide this by the Difference betwixt the Product of Annuity into the Square of the Time, and the Sum of the Product of Time and Annuity, and twice the Product of present Worth and Time; the Quote is the Rate.

For Annuities *payable in half yearly or quarterly Payments.*

In the preceding *Problems*, let t represent the Number of half Years or quarters that an Annuity continues; r the Interest of 1 *l.* for $\frac{1}{2}$ or $\frac{1}{4}$ of a Year; and n the $\frac{1}{2}$ Years or quarters Payment: Then all the preceding Rules are applicable to half yearly or quarterly paid Annuities the same way as to yearly Payments.

PART II. *Of* Annuities *at Compound Interest.*

Observe; *In the following* Problems, *the Amount of* 1 *l. and* 1 *Year's Interest is called* the Rate. *For Example:* 1.05, *if Interest is at* 5 per Cent.

1. *Of* Annuities *in* Arrears *at Compound Interest.*

PROB. 1. Having the Annuity, Rate and Time (in whole Years) to find the Amount. *Rule:*

Rule. Make 1 the least Term of a Geometrical Progression, the Rate the 2d Term, (which consequently is the Ratio of the Progression by which every Term is to be multiplied to produce the next) carry it to as many Terms as the Number of Years; its Sum is the Amount of 1 *l.* Annuity for the given Time, [and to find the Sum most easily, multiply the last Term by the Rate, or Ratio, which produces a Power of the Rate whose Index is the Time; and from the Product take 1 the 1st Term; then divide the Remainder by the Rate less 1; the Quote is the Sum] which Sum multiplied by the given Annuity, the Product is the Amount sought.

Example. What is the Amount of an Annuity of 40 *l.* to continue 5 Years, allowing Compound Interest at 5 *per Cent.*? Answer, 221.02525 *l.* which is 221 *l.* 0 *s.* 6.06 *d.* *Thus:* Of a Geometrical Progression beginning with 1, whose Ratio is 1.05, the 5th Term is $\overline{1.05}^4 = 1.21550625$, and the Sum of the Series is 5.52563125. For $\overline{1.05}^5 = 1.2762815625$, and $\overline{1.05}^5 - 1 = .2762815625$; which divided by $1.05 - 1$, or .05, the Quote is 5.52563125, the Amount of 1 *l.* Annuity for 5 Years. Which multiplied by 40, the Product is $221.02525 = 221$ *l.* 0 *s.* 6.06 *d.* the Amount of 40 *l.* Annuity.

DEMONSTR. It is plain that upon the 1st Year's Annuity there will be due as many Years Compound Interest, as the given Number of Years less 1; and gradually one Year less upon every succeeding Year to that preceding the last, which has but one Year's Interest, the last having no Interest due: But the Amount of 1 *l.* for 1 Year being 1.05, the Amounts of it for 2, 3, *&c.* Years are (by *Prob.* 1. *Part* 2. §. 1.) the several Powers or Products of 1.05 multiplied continually by itself 2, 3, *&c.* times; consequently $\overline{1.05}^4$, $\overline{1.05}^3$, $\overline{1.05}^2$, 1.05, 1 are the Amounts of the 1st, 2d, 3d, 4th and 5th Years Annuity of 1 *l.* whose Sum is therefore the whole Amount of the Annuity of 1 *l.* for 5 Years. But 1 *l.* is to the Amount of 1 *l.* as any other Annuity to its Amount. Wherefore the Amount of 1 *l.* *Ann.* multiplied by another *Ann.* gives its Amount.

Universally. Let R be the Rate, or Amount of 1 *l.* with 1 Year's Interest; then the Series of Amounts of several Years of 1 *l.* Annuity from the last to the first is 1, R, R^2, R^3, *&c.* R^{t-1}. And the Sum of this, according to the Rule of a Progression Geometrical, is $\frac{R^t-1}{R-1}$, the Amount of 1 *l.* Annuity for *t* Years. And this multiplied by any other Annuity *n* gives the Amount of that Annuity, *viz.* $\frac{R^t-1}{R-1} \times n$, or $\frac{nR^t-n}{R-1}$, (the universal Expression of the Rule) because all Annuities are proportional with their Amounts, and $1 : \frac{R^t-1}{R-1} :: n : \frac{R^t-1}{R-1} \times n$.

SCHOLIUM. As $1 : R : R^2$, *&c.* R^{t-1} is the Series of Amounts of 1 *l.* Annuity forborn for *t* Years; so is $n : nR : nR^2$, *&c.* nR^{t-1} the Amounts of the Annuity *n*. Wherefore the Amount of any Annuity for *t* Years is the Sum of a Geometrical Progression, whose least Term is the Annuity, and Number of Terms equal to the Years, the Ratio being the Sum of 1 *l.* and one Year's Interest expressed here by R. And by the Rule of Geometrical Progressions, the Sum of this Series is $\frac{nR^t-n}{R-1}$ as before.

Another Rule. Find a principal Sum, of which 1 Year's Interest is equal to the given Annuity; then find the Amount of that Principal for the given Time and Rate, (by *Prob.* 1. *Part* 2. §. 1.) from which Amount subtract the Principal, the Remainder is the Amount of the Annuity.

The *Reason* is plain; For the Amount of the Principal is the Sum of the Principal, and every Year's Simple Interest, (which make the several Years Annuities) with all the Compound Interest arising from these; so that the Principal taken from the Amount, leaves the

Sum of the several Years Simple Interest with all their Compound Interest; which is plainly the whole Amount of the Annuity.

In the preceding *Example*; I say, as .05 to 1, so is 40 to 800 a Principal, which gives 40*l.* Interest in 1 Year at 5 *per Cent.*; then the Amount of this Principal forborn 5 Years at Compound Interest is 1021.02525; from which take 800, the Remainder is 221.02525 as before.

SCHOLIUM. This *Rule* may be proved from its Coincidence with the former, thus; R being the Sum of 1 *l.* and 1 Year's Interest; then is $R-1$ the Year's Interest, and $R-1 : 1 :: n : \frac{n}{R-1}$ the Principal, whose Year's Interest is n. Then the Amount of this Principal for t Years is $\frac{nR^t}{R-1}$ (by *Prob.* 1. *Part* 2. §. 1.) from which take the Principal $\frac{n}{R-1}$, the Remainder is $\frac{nR^t-n}{R-1}$ the Amount by the former *Rule*.

PROB. 2. Having the Amount, Rate and Time, to find the Annuity.

Rule. Take any Annuity at pleasure, and find its Amount by the last. Then as that Amount is to its Annuity, so is the given Amount to its Annuity. And *observe*, If you chuse 1 *l.* Annuity, you have nothing to do but divide the given Amount by the Amount of 1 *l.* Annuity.

Example. What Annuity will amount to 221 *l.* 0 *s.* 6 *d.* in 5 Years at the Rate of 5 *per Cent.*? Answer, 40 *l.* For the Amount of 1 *l.* Annuity in 5 Years is 5.52563125; by which dividing the given Amount 221.025, the Quote is 39.999 = 39 *l.* 19 *s.* 11 *d.* 3 *f.* &c. or 40 *l.* nearly; which it would have been precisely, had we taken 221 *l.* 0 *s.* 6.06 *d.* (or 221.02525) for the Amount, as in the 1st *Problem.*

DEMONSTR. The Reason of this Rule is plain: For all Annuities and their Amounts must be proportional.

SCHOLIUM. The Amount of 1 *l.* Annuity is, by *Prob.* 1. expressed $\frac{R^t-1}{R-1}$; and calling the given Amount A, then this Rule is $n = A \div \frac{R^t-1}{R-1} = \frac{AR-A}{R^t-1}$; that is, multiply the Rate by the Amount, and from the Product take the Amount; the Difference divide by the Difference of 1, and the t Power of the Rate. The Use of this Expression of the Rule you'll find in the following *Problems*. And we may also deduce it thus, by *Prob.* 1. $A = \frac{nR^t-n}{R-1}$; hence $AR-A = nR^t-n$, and $\frac{AR-A}{R^t-1} = n$.

PROBL. 3. Having the Annuity, Rate, and Amount, to find the Time.

Rule. Find a corresponding Principal (as in the 2d *Rule* of *Probl.* 1.), the Sum of this and the given Amount, is the Amount of that Principal for the given Rate and Time sought: Wherefore find the Time (by *Probl.* 3. *Part* 2. §. 1.) thus; divide that Amount by its Principal, then multiply the Rate continually by it self till the Product or Power produced be equal to the former Quote, the Index of that Power, or Number of Multiplications more 1, is the Number of Years sought.

Example. In what Time will 40 *l.* Annuity amount to 221.02525 *l.* at the Rate of 5 *per Cent.*? Answer, 5 Years. Thus, .05 : 1 :: 40 : 800 the corresponding Principal, then $800 + 221.02525 = 1021.02525$; which divided by 800, the Quote is 1.2762815625, equal to the 5th Power of 1.05, or $\overline{1.05}^{5}$; so that 5 is the Number of Years sought.

DEMON-

DEMONSTR. By what's shewn in *Prob.* 1. *Rule* 2. the Difference of the Principal found and its Amount, is the Amount of the Annuity; wherefore that Principal added to the Amount of the Annuity gives the Amount of the Principal: The rest of the *Rule* is demonstrated in the *Problem* referred to.

SCHOLIUM. The corresponding Principal is expressed $\frac{n}{R-1}$ (as in *Schol.* to *Prob.* 1.) and this added to the given Amount A, the Sum is $A + \frac{n}{R-1} = \frac{n + AR - A}{R-1}$; which divided by the Principal $\frac{n}{R-1}$, the Quote is $\frac{n + AR - A}{n} = R^t$ by the Rule, and is another Expression of it; which may also be deduced thus, by *Prob.* 2. $\frac{AR - A}{R^t - 1} = n$; hence $AR - A = nR^t - n$, and $n + AR - A = nR^t$; and lastly, $\frac{n + AR - A}{n} = R^t$.

PROB. 4. Having the Annuity, Amount and Time, to find the Rate.

There is no Rule within my Limits that will solve this *Problem*, except for that one Case where the Time is 2 Years. And to come at that Rule, I must deduce the general Rule from what precedes, and so leave it with the Application to this Case.

In the last *Problem* we saw $AR - A = nR^t - n$. Whence $AR = nR^t - n + A$; and again, $AR - nR^t = A - n$; and dividing by n, it is $\frac{A}{n} \times R - R^t = \frac{A-n}{n}$. Now if t is above 2, we can make nothing of it by common Methods: But if t is $=2$, then it's $\frac{A}{n} \times R - R^2 = \frac{A-n}{n}$. And calling $\frac{A}{n} = d$, then (by *Prob.* 6. *Chap.* 2. *Book* III.) $R = \frac{d}{2} +$, or $- \overline{\frac{A-n}{n} - \frac{dd}{4}}|^{\frac{1}{2}}$.

2. *Of the Purchase of Annuities at* Compound Interest.

PROBL. 5. Having the Annuity, Rate, and Time, to find the Present Worth.

Rule 1. Find the Present Worth of each Year by itself (by *Probl.* 2. *Part* 2 §. 1.); the Sum of all these is the Present Worth sought.

Rule 2. Find the Amount of the Annuity (by *Probl.* 1.) then find the Present Worth of this Amount as a Sum due at the End of the whole Time (by *Probl.* 2. *Part* 2. §. 1.); it is the Present Worth sought.

Rule 3. Find a Principal Sum whereof the Annuity is 1 Years Interest; and find the Present Worth of it as a Sum due at the End of the Time; subtract this Present Worth from its Principal; the Remainder is the Present Worth of the Annuity.

Observe, The last *Rule* is the easiest, and therefore I apply it in the following *Example*; but there was a Necessity to deliver them all, because the 1st is the fundamental Rule, which has a Reason in it self; the 2d having its Reason only in its Coincidence with the 1st, as to the *Answer*; and the 3d depends upon the 2d.

Example. What is the Present Worth of an Annuity of 40 *l.* to continue 5 Years, discompting at 5 *per Cent.*? *Answer*: 173 *l.* 3 *sh.* 7 *d.* Found thus; as .05 to 1, so is 40 to 800, a Principal Sum whereof the Present Worth, discompting Compound Interest for 5 Years, at 5 *per Cent.* is 626 *l.* 16 *sh.* 5 *d.* [for the Amount of 1 *l.* in 5 Years is $\overline{1.05}|^5 = 1.276281$, *&c.* and $800 \div 1.276281 = 626.8212$, *&c.* $=$ 626 *l.* 16 *sh.* 5 *d.* nearly] then from 800 take 626 *l.* 16 *sh.* 5 *d.* the Remainder is 173 *l.* 3 *sh.* 7 *d.*

DEMON-

DEMONSTR. 1. That the 1st *Rule* gives the true *Answer* agreeable to the plain Meaning of the *Question*, is manifest, and is also confirm'd by what has been said upon the like Question with Simple Interest; so that every other Rule must coincide with this in the *Answer*, else it cannot be true; and that the other two *Rules* give the same *Answer*, I shall also demonstrate.

2. For the 2d Rule. The Present Worth of any single Year amounts to the Annuity when it becomes payable. [For *Example* · the Present Worth of the 3d Year is a Principal, which in 3 Years Time will make an Amount equal to the Annuity] and therefore the Amount of that Year from the Time it falls due to the End of the given Time, is the same as the Amount of the Present Worth of it from the Time of the Purchase to the End of the Annuity. Consequently, the Present Worth of any Year, discompting from the Time it falls due, is the same as the Present Worth of the Amount of that Year, summed up to the End of the Annuity, and then discompted to the Time of the Purchase. But the Amount of the Annuity at the End of the whole Years is the Sum of the Amounts of the several Years; and consequently the Present Worth of this Sum is the Sum of the Present Worths of these particular Amounts; which being equal to the Present Worths of the several Years discompted from the Times they fall due: Therefore both *Rules* give the same *Answer*.

3. For the 3d *Rule*, It follows from the 2d thus. The Principal found by the *Rule*, will make an Amount from the Time of the Purchase to the End of the Annuity, equal to the Sum of it self, and the Amount of the Annuity (as we saw in the last *Problem*); but the Present Worth of any Sum is equal to the Sum of the Present Worths of any two (or more) Parts of that total Present Worth; consequently, the Present Worth of one Part taken from the Present Worth of the Whole, leaves the Present Worth of the other Part; *that is*, in the present Case, The Present Worth of the Principal found (which is a Part of its own Amount) taken from the same Principal (which is the total Present Worth of its total Amount), leaves the Present Worth of the Annuity (which is the other Part of the Amount of that Principal).

SCHOLIUM. By *Probl.* 1. the Amount of the Annuity is expressed thus, $A = \frac{nR^t - n}{R - 1}$ and (by *Probl.* 2. *Part* 2. §. 1.) the Present Worth of this is $p = \frac{nR^t - n}{R^{t+1} - R^t}$, which is therefore the Expression of the 2d *Rule*: And that the 1st *Rule* resolves into the same Expression, may be thus shewn: The Present Worths of the several Years are (by *Probl.* 2. *Part* 2. §. 1.) thus expressed, $\frac{n}{R} : \frac{n}{R^2} : \frac{n}{R^3}$, *&c.* to $\frac{n}{R^t}$, which is a Geometrical Progression, in the Ratio of R to 1, whose Sum is, by the Rule of a Geometrical Progression, $\frac{nR^t - n}{R^{t+1} - R^t}$, as before. The Coincidence of the 3d *Rule* with this may also be easily shewn after the same Manner; wherefore, instead of the preceding three *Rules*, we may take it according to this universal Expression, thus;

Take two Powers of the Rate whose Indexes are the Time, and the Time more 1; multiply the 1st of these Powers by the Annuity, and take the Annuity from the Product; then divide this Difference by the Difference of these two Powers; the Quote is the Present Worth.

Or thus: Take the Difference of 1 and the lesser Power, which divide by the Difference of the Powers, and then multiply the Quote by the Annuity, because $\frac{nR^t - n}{R^{t+1} - R^t} = \frac{R^t - 1}{R^{t+1} - R^t} \times n$.

COROLL. Hence, knowing the Time any Annuity continues, and the Rate allowed in the Purchaſe, we can find how many Years and Parts of a Year's Purchaſe it is worth, without knowing the Annuity: For ſince the Price is $\frac{R^t - 1}{R^{t+1} - R^t} \times n$, therefore, whatever n is, $\frac{R^t - 1}{R^{t+1} - R^t}$ does univerſally expreſs the Number of Times the Annuity is contained in the Price.

PROBL. 6. Having the Preſent Worth, Rate, and Time, to find the Annuity.

Rule. Take any Annuity at pleaſure, and find its Preſent Worth by the laſt; then, as that Preſent Worth is to its Annuity, ſo is the given Preſent Worth to the Annuity ſought. *Obſerve,* If you take 1 *l.* Annuity, then there is no more to be done but divide the given Preſent Worth by that of 1 *l.* Annuity.

Example: What Annuity, to continue 5 Years will be purchaſed for 173 *l.* 3 *ſh.* 7 *d.* allowing Compound Intereſt at 5 *per Cent.*? *Anſwer:* 40 *l.* Found thus:

The Preſent Worth of 1 *l.* Annuity to continue 5 Years is 4.329, and 173 *l.* 3 *ſh.* 7 *d.* or 173.1788, *&c.* ÷ 4.329 = 40.004, *&c.* which is 40 *l.* nearly, for the Decimal is leſs than a Farthing.

DEMONSTR. The Reaſon of this *Rule* is plain: For all Annuities are proportional with their Preſent Worths for the ſame Time and Rate.

SCHOLIUM. By *Schol.* to the preceding *Problem,* the Preſent Worth of 1 *l.* Annuity is expreſſed $\frac{R^t - 1}{R^{t+1} - 1}$; wherefore the Annuity anſwering to any Preſent Worth p (for the ſame Time and Rate) is $p \div \frac{R^t - 1}{R^{t+1} - 1} = p \times \frac{R^{t+1} - 1}{R^t - 1}$, which is an univerſal Rule. We may alſo deduce it from the preceding *Problem,* thus; $p = \frac{nR^t - n}{R^{t+1} - 1}$, whence $pR^{t+1} - p = nR^t - n$, and dividing by $R^t - 1$, it is $n = \frac{pR^{t+1} - p}{R^t - 1}$, or $p \times \frac{R^{t+1} - 1}{R^t - 1}$.

PROBL. 7. Having the Annuity, Preſent Worth, and Rate, to find the Time.

Rule. Find a Principal whereof 1 Year's Intereſt is the Annuity, from which ſubtract the given Preſent Worth; the Remainder is the Preſent Worth of that Principal conſidered as a Sum due at the End of the Annuity: Therefore, by *Probl.* 3. *Part* 2. §. 1. find in what Time this Preſent Worth will amount to the Principal found; it is the Time ſought.

Example: What Time muſt an Annuity of 40 *l.* continue, to be worth of ready Money 173 *l.* 3 *ſh.* 7 *d.* allowing 5 *per Cent.* Compound Intereſt. *Anſwer:* 5 Years.

For 05 : 1 :: 40 : 800 the Principal ſought; from which take 173 *l.* 3 *ſh.* 7 *d.* the Remainder is 626 *l.* 16 *ſh.* 5 *d.* or 626.82 *l.* nearly, the Preſent Worth of 800 due at the End of the Time ſought; then $800 \div 626.82 = 1.27678$, *&c.* $= \overline{1.05}^5$ nearly, (for this is 1.276281) ſo the Time ſought is 5 Years.

DEMONSTR. It was ſhewn in *Probl.* 5. that the Difference of the Principal found and its Preſent Worth, is the Preſent Worth of the Annuity; conſequently the Difference of

of the Present Worth of the Annuity, and the same Principal, must be the Present Worth of that Principal. The rest of the *Rule* is demonstrated in the *Problem* referred to.

SCHOLIUM. The corresponding Principal being expressed $\frac{n}{R-1}$ (for $R-1:1::n:\frac{n}{R-1}$) and the given Present Worth p, then by this Rule $\frac{n}{R-1}-p=\frac{n+p-pR}{R-1}$ is the present Worth of $\frac{n}{R-1}$: And by *Probl.* 3. *Part* 2. §. 1. it is $\frac{n}{R-1}\div\frac{n+p-pR}{R-1}$ $\left(=\frac{n}{n+p-pR}\right)=R^t$, which is the universal Expression of the Rule; to be deduced also from *Probl.* 5. thus, $p=\frac{nR^t-n}{R^{t+1}-R^t}$; hence $pR^{t+1}-pR^t=nR^t-n$, and $pR^{t+1}=nR^t-n+pR^t$, then $pR^{t+1}+n=nR^t+pR^t$; also $n=nR^t+pR^t-pR^{t+1}$ $=\overline{n+p-pR}\times R^t$; and lastly, $\frac{n}{n+p-pR}=R^t$.

PROBL. 8. Having the Annuity, Present Worth, and Time, to find the Rate.

This *Problem* can be solved by no Method within my Limits, except for that Case where the Time is 1 Year; which I shall shew by deducing the general Rule from the preceding; thus, in the *Scholium* to the last *Problem* we see $n=nR^t+pR^t-pR^{t+1}=$ $\overline{n+p}\times R^t-pR^{t+1}$, and $\frac{n}{p}=\frac{n+p}{p}\times R^t-R^{t+1}$; now if t is any Number greater than 1, to find R we have an affected Root, above the Square, to extract; for which we have no Rules in this Work: But if the Time is 1 Year, then the Rule is $\frac{n}{p}=\frac{n+p}{p}\times R-$ R^2; call $\frac{n+p}{p}=d$, and then, by *Probl.* 6. *Ch.* 2. *Book* III. $R=\frac{d}{2}+\text{ or }-\overline{\frac{n}{p}-\frac{dd}{4}}\Big|^{\frac{1}{2}}$

Of Annuities *payable Half-yearly or Quarterly, and other Questions, wherein the Time is some Part of a Year, or whole Years with Part of a Year.*

Part I. IF Annuities are payable in Half-years or Quarters (the Half-year or Quarter's Payment being the $\frac{1}{2}$ or $\frac{1}{4}$ of the whole Years Payment), then if in the preceding *Problems* t express the Number of Half-Years or Quarters, and n the Half-Year's or Quarter's Payment, also R the Sum of 1 *l.* with $\frac{1}{2}$ or $\frac{1}{4}$ of a Year's Interest of it, taken according to any Supposition (*i. e.* By either of the *Methods* explained in *Schol.* 2. *Probl.* 1. *Part* 2. §. 1. and here the 2d *Method* seems to be the fairest), then all the preceding *Rules* are applicable to Half-yearly and Quarterly Payments the same way as for Years; yet the Labour of the Calculation is hereby increased.

Part II. But whether Annuities are payable Yearly or Quarterly, the Time sought may perhaps not come out in whole Years or Quarters precisely; which is a Sign that the Question is impossible in whole Years or Quarters: So that if what is wanting in the Number found, to make it whole Years or Quarters, be very little, it may be neglected; but if it is great, then we may find some Part of the Year or Quarter's Payment corresponding to it, as

I ſhall briefly explain with reſpect to Annual Payments; and, to make this Part compleat, I ſhall alſo conſider thoſe Queſtions wherein the Time is given in Years and Parts of a Year. And *obſerve* too, That tho' the Payments are by Half-years or Quarters, the Calculations may juſtly be made as if they were by Years.

1. In *Probl.* 1. If there is any Part of a Year, then, having found the Amount for the whole Years, find the Intereſt of that Amount for the Parts of a Year in the Queſtion, by any of the *Methods* explained in *Schol.* 2. *Probl.* 1. *Part* 2. §. 1. as you think fit, and add that to the Amount for the whole Years.

2. In *Probl.* 2. If the Time is whole Years and Parts of a Year, the Solution depends upon the laſt.

3. In *Probl.* 3. Where the Time is ſought, the laſt Part of the Solution depends upon *Probl.* 3. *Part* 2. §. 1. And for the Parts of a Year in the Time ſought, ſee the *Schol.* to that *Problem.*

4. For *Probl.* 4. we can make nothing of it; and it is indeed of little Uſe.

5. In *Probl.* 5. if the Time is whole Years and Parts of a Year, the Solution will be different according as you ſuppoſe Intereſt for the Parts of a Year to be taken. But however that be, this is firſt to be done, *viz.* to find the Preſent Worth, diſcompting for the Whole Years in the Queſtion; then find the Preſent Worth of this Preſent Worth, diſcompting for the Parts of a Year in the Queſtion. See *Schol.* 3. to *Probl.* 2. *Part* 2. §. 1.

6. For *Probl.* 6. the Solution depends upon the laſt.

7. For *Probl.* 7. the laſt Part of the Solution depends upon *Probl.* 3. *Part* 2. §. 1. See the *Schol.* to that *Problem.*

8. For the 8. *Probl.* we can make nothing of it; and it's of as little Uſe as the 4th.

§. III. *Of the* Purchaſe *of* Freehold Eſtates, *or* Annuities *to continue for ever, allowing Compound Intereſt.*

PROBL. TO find the Price or Preſent Worth of a Freehold Eſtate or Annuity to continue for ever, diſcompting Compound Intereſt.

Rule. Find a Principal Sum of which one Year's Intereſt is the Rent or Annuity given; it is the Price ſought.

Example: What is the Price of a perpetual Annuity of 40 *l.* diſcompting at 5 *per Cent.* Compound Intereſt? *Anſwer:* 800 *l.* for .05 : 1 : : 40 : 800 *l.*

DEMONSTR. The *Reaſon* of this *Rule* ſeems of it ſelf obvious; for it's plain, that ſince a Year's Intereſt of the Price is the Annuity, therefore there can neither more nor leſs be made of that Price than of the Annuity, whether we employ it by Simple or Compound Intereſt.

SCHO-

SCHOLIUMS.

1. But leſt any body ſhould doubt whether the Diſcompt is here made at Simple or Compound Intereſt, let them conſider the *Rule* of *Probl.* 2. *Part* 2. §. 1. for the Preſent Worth of an Annuity, which is thus expreſſed, $p = \frac{n}{R} + \frac{n}{R^2} + \frac{n}{R^3}$, *&c.* (*n* being the yearly Rent or Annuity). Now if the Annuity continues for ever, then this Geometrical Progreſſion goes on, decreaſing for ever; and the Sum of it, by *Probl.* 1. *Ch.* 3. *Book* V. is $\frac{n}{R-1}$, which is a Principal Sum of which one Year's Intereſt is the Annuity *n*; for $R-1:1::n:\frac{n}{R-1}$.

2. Again: Of theſe three things, the Annuity to continue for ever, the Price, and Rate of Intereſt, any two of them being given, the third may be found eaſily, from the preceding *Rule*, which being expreſſed thus, *viz.* $p = \frac{n}{R-1}$, hence it follows that $n = pR - p$, and $R - 1 = \frac{n}{p}$, or $R = \frac{n}{p} + 1$.

3. The Unreaſonableneſs of purchaſing Annuities at Simple Intereſt is further confirmed by this *Problem*: For the Price of a perpetual Annuity, diſcompting Simple Intereſt, as it would be greater than when Compound Intereſt is diſcompted, it would therefore be a Principal Sum of which 1 Year's Intereſt is greater than the Annuity: And indeed the Price would be infinitely great, or greater than any aſſignable Number.

§. IV. *Of* Annuities, *or* Leaſes *in* Reverſion.

HItherto I have conſidered Annuities as immediately entered upon, *i. e.* That the 1ſt Year commences preſently or at payment of the Money: But if it is in *Reverſion*, then we muſt conſider the Time betwixt payment of the Money and the beginning of the 1ſt Year of the Annuity; and therefore,

1. In *Probl.* 5. §. 3. having found the Preſent Worth as if the Annuity were immediately entred upon, find again the Preſent Worth of that Preſent Worth, rebating for the Time betwixt the Purchaſe and Commencement of the Annuity; and this is the Anſwer.

2. In *Problems* 6. and 7. take the given Preſent Worth, and find its Amount for the Time of *Reverſion*, and take this Amount as the Preſent Worth paid at the Commencement of the Annuity, and proceed with it.

Obſerve, Annuities, and Rents of Houſes or Lands, are of the ſame Nature, wherein the ſame Queſtions occur as to their being in Arrears or being purchaſed: But with reſpect to *Leaſes* there ariſe Queſtions with ſome different Circumſtances, owing to the Practice of taking what they call *Fines*, which is a Sum of Money paid at the Beginning of the Leaſe, beſides the Yearly Rent. I ſhall give you a few of theſe Queſtions, with a general Direction for the Solution; which will be ſufficient upon this Subject.

Queſtion 1. There is a Piece of Land worth 20 *l.* yearly Rent, and 100 *l.* of Fine for a Leaſe of 21 Years: The Maſter is willing to quit the Fine, and increaſe the Rent; What

What ought the Rent to be? *Rule.* Find what Rent or Annuity, to continue 21 Years, 100 *l*, will purchase, discompting at the agreed Rate of Interest; the Sum of that and the former Rent is the Rent sought. *Observe,* If the whole Fine is not to be taken away, find the Annuity answering to the Part taken away.

Quest. 2. A Piece of Land is worth 12 *l.* Yearly Rent, and a Fine of 30 *l.* for 19 Years: The Farmer is willing to pay more Fine, and reduce the Rent to 20 *l.* What ought the Fine to be? *Rule.* Take the Difference of the two Rents (30 and 20), and find the Present Worth of an Annuity equal to the Difference for the same Time (19 Years); that is the additional Fine to be paid. *Observe,* The same way the whole Rent may be taken away.

Quest. 3. There is a Farm to be let for 21 Years, at 10 *l.* Yearly Rent and 20 *l.* of Fine; if the same be lett for 30 Years at the same Rent, What ought the Fine to be? *Rule.* Find what Annuity 20 *l.* will purchase for 21 Years, at the agreed Rate (that added to the Rent 10 *l.* is the true Rent with no Fine), therefore find the Present Worth of that Annuity to continue 30 Years; it is the Fine sought.

Quest. 4. A Person has 7 Years to run of a Lease of 21 Years, for which he paid 40 *l.* Fine and 15 *l.* Yearly Rent: He would renew the Lease to 16 Years from this Time (*i. e.* for 12 Years after the first Lease expires): What Fine ought he to pay? *Rule.* Find what Rent for 21 Years the given Sum 40 *l.* will purchase; then find the Present Worth of this Yearly Rent to continue 12 Years; lastly, find the Present Worth of this last Present Worth, rebating for 7 Years, the Time that remains of the old Lease; this is the Fine to be paid.

General SCHOLIUM, *with an* Abstract *of the preceding* Problems *where Compound Interest is concerned.*

It is thought the most convenient Method for the Calculation of Annuities, to have *Tables* ready made, extending to the greatest Number of Years that ordinarily occur in that Business, and for several Rates of Interest that are most likely to occur; by Means of which the Answers of the more useful of the preceding *Problems* may be easily found.

The *Tables* which are common upon this Subject are limited to 1 *l.* Thus, we have 1. A Table of the Amounts of 1 *l.* for 30 or 50 Years, at several Rates Compound Interest. 2. Of the Present Worths of 1 *l.* due after any Number of Years from 1 to 30 or 50. 3. The Amounts of 1 *l.* Annuity. 4. The Present Worths of 1 *l.* Annuity. 5. The Annuity to be purchased for 1 *l.* By Means of which the Answer to any of the preceding *Problems* may be easily got for any other Sum than 1 *l.* by one Multiplication or Division.

But here I must *observe,* That those who are employed in such Calculations ought to understand the Rules at large, and so be able to examine and make Tables for themselves: And in my Opinion, it is not fit in Questions of Consequence to trust to any Tables but what one has examined, or made for himself. Now all that is proposed by the Tables is the Ease and Expeditiousness of the Calculation; and the greatest Burden of the Work is the finding the Power or Product of the Rate, (or Sum of 1 *l.* and 1 Year's Interest) multiplied continually into itself, as the Rules direct; which, if the Number of Years in the Question is great, becomes very tedious; for this being found, the Rest of the Work, in the more useful of the preceding *Problems,* is no more than one or two Operations of Multiplication or Division; with a simple Addition or Subtraction in some Cases: And therefore, whoever can readily do Multiplication or Division with Integers or Decimals (and no other can perform these Calculations, even with the common Tables, which require at least one Multiplication or Division) needs no more than a Table of the Powers of

of several Rates of Interest carried to a convenient Length; which are the Amounts of 1 *l.* Principal for 1, 2, *&c.* Years at Compound Interest (for I speak of no other here), according to the *Rule* of *Probl.* 1. §. 1. *Part* 2. The following *Table* at 5 *per Cent.* to 31 Years is an Example which may be extended to more Years, and the like made for other Rates of Interest. Whoever understands the preceding *Rules* will know the Uses to be made of this *Table*; yet it will be convenient along with this *Table* to point out the Use of it (for it will be needless to make *Examples*, since the preceding are sufficient), by which you'll have the preceding *Rules* drawn together in one short and clear View.

TABLE *of the Powers of* 1.05, *or the* Amounts *of* 1 l. *at* 5 per Cent.

Yrs.		
1	1.05 *l.*	= R the Rate.
2	1.1025	$= R^2$
3	1.157525	$= R^3$
4	1.215506	$= R^4$
5	1.276281	$= R^5$
6	1.340096	$= R^6$
7	1.407100	$= R^7$
8	1.477455	$= R^8$
9	1.551328	$= R^9$
10	1.628895	$= R^{10}$
11	1.710339	$= R^{11}$
12	1.795856	$= R^{12}$
13	1.885649	$= R^{13}$
14	1.979932	$= R^{14}$
15	2.078928	$= R^{15}$
16	2.182874	$= R^{16}$
17	2.292018	$= R^{17}$
18	2.406919	$= R^{18}$
19	2.526950	$= R^{19}$
20	2.653298	$= R^{20}$
21	2.785962	$= R^{21}$
22	2.925261	$= R^{22}$
23	3.071524	$= R^{23}$
24	3.225100	$= R^{24}$
25	3.386355	$= R^{25}$
26	3.555673	$= R^{26}$
27	3.733456	$= R^{27}$
28	3.920129	$= R^{28}$
29	4.116135	$= R^{29}$
30	4.321942	$= R^{30}$
31	4.538039	$= R^{31}$

The Use *of this* Table *in the Solution of the preceding* Problems.

1. *Of* Interest.

1. To find the Amount of any Principal for any Time, at 5 *per Cent.* Compound Interest. *Rule.* Take the tabular Number against the Number of Years, its Product by the Principal gives the Amount, which, in *Probl.* 1. §. 1. *Part* 2. is expressed thus, $A = p \times R^t$.

2. To find the Principal (or Present Worth) answering to any Amount (or Debt payable) after certain Years. *Rule.* Divide the Amount by the tabular Number against the Number of Years; the Quote is the Principal; expressed thus, $p = \frac{A}{R^t}$, as in *Probl.* 2. §. 1. *Part* 2.

3. To find in what Time any Principal will make a certain Amount. *Rule.* Divide the Amount by the Principal, and seek the Quote, or the next lesser Number in the *Table*, and against it is the Number of Years sought. Because $\frac{A}{p} = R^t$, as in *Probl.* 3. §. 1. *Part* 2.

Observe, For finding the Rate from the Principal Amount and Time, we should have Tables at different Rates of Interest; then taking the Quote of the Amount, divided by the Principal, seek it against the Number of Years in the Tables: The Rate of that Table where it is found is the Rate sought. But if it is not exactly found, take the next Number lesser and also the next greater (if both a greater and lesser canbe found among the Tables) and the Rates of these two *Tables* are Limits betwixt which the Rate sought lies.

2. *Of* Annuities.

1. To find the Amount of an Annuity. *Rule.* From the tabular Number against the Number of Years take 1, and divide the Remainder by the Rate less 1; multiply the Quote by the Annuity; the Product is the Amount; expressed thus, $\frac{R^t - 1}{R - 1} \times n = A$, as in *Probl.* 1. §. 2. *Part* 2.

2. To

2. To find the Annuity which makes a certain Amount. *Rule.* Take the Quote directed in the last *Rule*, and by it divide the Amount; this Quote is the Annuity sought. thus, $A \div \frac{R^t - 1}{R - 1} = n$, as in *Probl.* 2. §. 2. *Part* 2.

3. To find in what Time any Annuity will make a certain Amount. *Rule.* Divide the Annuity by the Rate less 1, to the Quote add the Amount, and divide the Sum by the first Quote; seek this last Quote, or the next lesser Number in the *Table*; against it stands the Number of Years. Thus, $\overline{\frac{n}{R-1} + A} \div \frac{n}{R-1} = R^t$, as in *Probl.* 3. §. 2. *Part* 2.

4. To find the Present Worth of an Annuity. *Rule.* Divide the Annuity by the Rate less 1; this Quote divide by the tabular Number against the Years, and subtract this last Quote from the first Quote; the Remainder is the Present Worth sought. Thus, $\frac{n}{R-1} - \frac{n}{R-1} \div R^t = p$, as in *Rule* 3. *Probl.* 5. §. 2. *Part* 2.

5. To find what Annuity a given Sum will purchase for certain Years. *Rule.* Take 1 from the tabular Number against the Years; divide it by the Difference betwixt that tabular Number and the next greater, and by this Quote divide the Present Worth; this last Quote is the Annuity sought, as in *Schol.* to *Problem* 6. §. 2. *Part* 2. thus expressed, $n = p \div \frac{R^t - 1}{R^{t+1} - R^t}$.

6. To find what Time an Annuity must continue to be worth a certain Price. *Rule.* Divide the Annuity by the Rate less 1, and from the Quote take the Price; by this Remainder divide the first Quote, and seek this last Quote in the *Table*; against it is the Number of Years. Thus, $\frac{n}{R-1} \div \overline{\frac{n}{R-1} - p} = R^t$, as in *Schol.* to *Probl.* 7. §. 2. *Part* 2.

Observe, That the preceding *Table* was made by compleating the Multiplication at every Step, and then taking only the first 6 decimal Places. Also, when the Figure in the 7th Place exceeded 5, 1 was added to the 6th Place, which makes the Error less, only it makes it Excessive instead of Defective.

§. V. *Of the* Equation of Payments.

DEFIN. WHEN several Debts are payable at several Terms (bearing no Interest till after the Term of Payment), to find a Term at which if they are all paid neither Debtor or Creditor loses any thing, is called, *Equating* the Terms of Payment, *i. e.* reducing them to one.

PROBL. To find the *equated* Term at which several Debts payable at different Terms, may be paid at once, without Loss to Debtor or Creditor, allowing Simple Interest.

For the Solution of this *Problem* there are different Methods, which I shall explain and compare.

Method 1. The Debts being expressed in Periods and decimal Parts, the Times in Years and Decimals, or Months and Decimals, if there are no Years, multiply each Debt by its Time; add all these Products into one Sum, and divide it by the Sum of the Debts; the Quote is the Time sought [reckoned from the Day of this Calculation, as the other Times are also supposed to be]. Exam-

Example: If 40 *l.* is due after 6 Months, 70 *l.* after 4 Months, the equated Time is 4 Months 21 Days: Found thus; $40 \times 6 = 240$. $70 \times 4 = 280$. $40 + 70 = 110$. $240 + 280 = 520$. then $520 \div 110 = 4.72$, *&c.* which is 4 Months 21 Days nearly, allowing 30 Days to a Month.

This *Rule* has Mr. *Cocker* and Mr. *Hatton* (among others) for its Defenders; who endeavour, each in his own way, to demonſtrate the Truth and Juſtice of it. How they have ſucceeded, I'll here examine.

Mr. *Hatton* ſuppoſes that the Intereſt of the Sum of the Debts from the Time of the Queſtion to the equated Time, ought to be equal to the Sum of the Intereſts of the ſeveral Debts from the Time of the Queſtion to their ſeveral Terms of Payment; and then, by an Example, ſhews that the Rule agrees to this Suppoſition. But this, inſtead of being a *Demonſtration*, is a plain begging of the Queſtion; for the Equity of that Suppoſition ought to have been firſt proved; againſt which there lies this obvious Reaſon, that the Debts bearing no Intereſt till after their Terms of Payments, the Conſideration of theſe Intereſts ſeems to be out of the Queſtion; and therefore this Suppoſition cannot be granted without Demonſtration. I ſhall preſently ſhew you other Reaſons againſt it; and in the mean time *obſerve*, that tho' it were right, yet Mr. *Hatton* has not demonſtrated his Rule; becauſe it is not enough to ſhew that a Rule gives a true Anſwer in one particular Caſe; a Demonſtration muſt comprehend all Caſes; and that the Rule is univerſally agreeable to the Suppoſition, I thus demonſtrate.

Take A for a Debt payable at the End of the Time t, and B for a Debt payable after the Time u: The equated Time muſt lie betwixt the two, and, by this *Rule*, it is $\frac{At+Bu}{A+B}$. Now the Intereſt of A for the Time t, is, (by *Probl.* 1. §. 1. *Part* 1.) Atr (r being the Rate of Intereſt for 1 *l.* in 1 Year, or Month, as the given Time t is). The Intereſt of B for the Time u is Bur, and the Sum of both is $Atr+Bur$. Alſo the Intereſt of $A+B$ for the equated Time $\frac{At+Bu}{A+B}$, is $\overline{A+B} \times r \times \frac{At+Bu}{A+B} = r \times \overline{At+Bu} = Atr+Bur$, as before: Therefore the Rule is univerſally agreeable to the Suppoſition.

Mr. *Cocker* ſuppoſes that the equated Time is right, when the Sum of the Intereſts of the ſeveral Debts that are payable before the equated Time, from their Terms to that Term, is equal to the Sum of the Intereſts of the Debts payable after the equated Time, from that Time to their Terms. The Agreement of the Rule with this Suppoſition, will eaſily appear by comparing it with the preceding Suppoſition: For they are manifeſt Conſequences one of another; and therefore the Rule that agrees to one muſt alſo agree to the other. Or we may eaſily prove the Agreement of the Rule with this laſt Suppoſition, after the ſame manner as I have done the former; but this I ſhall paſs over, and demonſtrate the Injuſtice of the Suppoſition, which Mr. *Cocker* endeavours to prove to be right by this Argument, *viz.* That what is gained by keeping ſome of the Debts after they are payable, is loſt by paying of others before they are due: But this is falſe; for tho' by keeping a Debt unpaid after it is due there is gained the Intereſt of it for that Time (becauſe the Creditor has a juſt Demand for that Intereſt), yet by paying a Debt before it is due the Payer does not loſe the Intereſt for that Time, but only the Diſcompt he could juſtly demand, which is leſs than the Intereſt (as former Rules ſhew). Thus we have a ſecond and more poſitive Demonſtration of the Error of this Rule. And the Fallacy in this laſt way of founding the Rule does plainly lead us to the true Foundation for a Solution of the *Problem*. But the explaining of that I ſhall leave to the laſt, and firſt explain the Methods of ſome other Authors.

Method 2. Mr *Kersey* was the first, as I know, who observed the Error of the preceding Rule, and has given us another in its place, with a more probable Foundation, tho' still wrong.

The Substance of Mr. *Kersey*'s Argument against the former Rule, is this, That there being no Consideration had of different Rates of Interest, the Answer will still be the same, whatever Rate be supposed; which he affirms to be wrong for this Reason, That to equal the Time so as neither Debtor or Creditor be a Loser, does imply some Rate of Interest; since otherwise any Day may be assigned for one entire Payment. His Meaning I take to be this, *viz.* that the Loss which either Party can sustain, must arise from the Interest of Money paid before or after it is really due; so that if there is no such thing as Interest, there can be no Loss, at whatever Time the Whole is paid; and if the Loss depends upon the Consideration of Interest, then different Rates of Interest must have different Effects, and occasion different Solutions to the Question: Therefore, concludes he, if some Rate of Interest be implyed, Equity requires that the Present Worth of the total Sum payable at one entire Payment, Discompt being made at that Rate of Interest, be equal to the Sum of the Present Worths of the particular Debts, discompting at the same Rate; upon which Foundation he gives us this Rule.

Kersey's Rule. Find the Present Worth of each Debt, discompting from the Time at which it is payable; then find in what Time the Total of these Present Worths would amount to the Total of the Debts; that is the Time sought.

Example: Suppose 300 *l.* due after 4 Months, 100 *l.* after 6 Months, and 100 *l.* after 12 Months: The equated Time discompting 6 *per Cent.* is 5.952, *&c.* Months. Found thus: Present Worth of 300 *l.* for 4 Months is 294.117, *&c. l.*; of 100 *l.* after 6 Months it is 97.087, *&c. l.*; and of 100 *l.* after 12 Months it is 94.339, *&c.* the Total is 485.544, *&c.* which will amount to 500 *l.* in 5.952, *&c.* Months, which is a little less than the Answer the preceding *Rule* would give.

The Argument for this *Rule* may be formed thus: If the Total of the Present Worths presently paid, is equivalent to the Sum of all the Debts paid at once, at the Time found, or also to the Sum of these Debts paid severally at their Terms, then these two last Methods of making Payment must be equal: Consequently the *Rule* is true, which is founded upon that Equality.

Now tho' this *Rule* has a more apparent and probable Ground of Equity than either of these alledged for the former, if the Rule for the Discompt of Simple Interest be allowed to be just and true; yet the Foundation is not strictly good, as will fully appear from the last *Method* after explained, whose Foundation is unexceptionable.

In the mean time, I make this Objection to it, *viz.* That the Answer will be different, if the Question is proposed at different Times, before the Term at which the first payable Debt falls due; *i. e.* if you alter all the given Times, so as there be the same Distances or Differences among them. Now this is a manifest Sign that the Rule is not good; because there is no Reason in the Nature of the thing, why the Answer should be different. For *Example*; One Debt is payable 3 Years hence, and another 7 Years hence: I suppose again, that the Calculation of an equated Time for the same Debts is proposed 2 Years after this (so that the one Debt is then payable after 1 Year, and the other after 5 Years), it is plain that the equated Term ought to fall upon the same Day as it would have done in the other Case (*i. e.* the Time last found must be 2 Years less than before), because the Debtor and Creditor continue in the same Situation to one another, neither of them gaining or losing by the two Years that are past since the Question was formerly proposed, there being nothing payable all that Time; so that the Determination of the *Problem* depends only upon the Consideration of the Distances of Time among the Terms of Payment of the several Debts: The Consideration of any Present Worths, except of those Debts whose Terms are beyond the equated Time, being out of the Question.

Method 3. Sir *Samuel Moreland,* after his pretended Correction of *Kerſey's* Rule for the Preſent Worth of Annuities at Simple Intereſt, gives us, in conſequence of that, another Rule than *Kerſey's* for the Equation of Payments; which is this;

Rule. Find the Amount of each Debt, with Intereſt from its Term of Payment to the Term at which the Debt laſt payable falls due; then find in what Time the Sum of all the Debts will amount to the Sum of the Amounts found; the Difference of that Time and the Time to which theſe Amounts were carried, is the Time ſought.

As to this Rule, I *obſerve,* 1. That, being founded on his Reaſoning about Preſent Worths of Annuities, which I have already ſhewn to be falſe, it muſt therefore alſo be falſe; and indeed more ſo than *Kerſey's* Rule.

But, 2. *obſerve,* That tho' it is differently expreſſed, yet it is in effect the very ſame as the 1ſt *Rule*; as will eaſily appear by comparing it with *Hatton's* Way of founding that *Rule.* Or you may alſo prove its Coincidence with that *Rule* after the ſame Manner as I have done the Agreement of *Hatton's* Suppoſition with it. Theſe things I leave to your own Exerciſe.

Before I explain the true Method of ſolving this *Problem,* I muſt take notice of what Mr. *Ward,* in his *Key to Intereſt,* has ſaid upon it. After he had mention'd the firſt *Rule* or *Method* above explained, he ſays,

" I ſhall paſs over all the Arguments made uſe of, *pro* and *con,* by Mr. *John Kerſey* and " Sir *Samuel Moreland,* and other Authors, about the Erroneouſneſs of this Rule, as alſo " the Rules they lay down inſtead of it, and ſhall only proceed to ſhew how the true " equated Time may be found from what hath been already done and proved."

And his Rule is this: Find the Diſcompts of all the particular Debts ſeparately, according to the Diſtances of their Terms of Payments; divide their Sum by the Product of the Preſent Worth of all the Debts, multiplied by the Rate of Intereſt (or Intereſt of 1 *l.* for 1 Year); the Quote is the equated Time.

Now from what is premiſed we had good Reaſon to expect a new Rule, quite different in its Effects from either *Kerſey's* or *Moreland's*; and yet its the very ſame in effect as *Kerſey's,* only differently expreſſed; which may be eaſily demonſtrated. But this I leave alſo to your own Exerciſe, and come to the true Solution.

Method 4. I come now to explain a Method, which no body, as I know, has hitherto publiſhed, and which has ſtrict Equity upon its Side, as far as Simple Intereſt has it in any Caſe.

To explain this Method, we muſt reflect upon the Reaſon brought againſt *Cocker's* Foundation of the 1ſt *Rule,* which was this, *viz.* That the Sum of the Intereſt of Debts kept in the Debtors Hands after they are due to the equated Time, and the Sum of the Intereſts of Debts paid before they are due from the equated Time to the ſeveral Terms, is not a juſt Balance of Gain and Loſs: Becauſe, tho' the Debtor gains the Intereſt of thoſe he keeps after they are due, yet he loſes only the Diſcompt upon thoſe he pays before they are due, which is leſs than the Intereſt; and therefore the Creditor has a juſt Exception to this Rule. Hence we ſee that as the true Foundation upon which the Time can be equated, is an Equality of Gain and Loſs, ſo the Gain conſiſts in the *Intereſts* of the Debts kept after they are due, and the Loſs in the *Diſcompts* of thoſe that are paid before they fall due: Wherefore, ſuch a Time muſt be found, as that the Sum of the Intereſts of the Debts due before the equated Time from their Terms to that Time, be equal to the Sum of the Diſcompts of thoſe Debts that are payable after the equated Time, from this to their ſeveral Terms. For which this is the

Rule. 1. Suppoſe two Debts: Then expreſſing the Debts in one Denomination (as *l.* and decimal Parts), find 1 Year's Intereſt of the Debt that is firſt payable (expreſſed in the

ſame Denomination) by which divide the Sum of the Debts, and add the Quote to the Sum of the Times, (taken in Years and decimal Parts, from the time of the Queſtion to the Terms of Payment of the two Debts): Call this the 1ſt Number found, Again, multiply each Debt by its Time, and divide the Sum of the Products by 1 Year's Intereſt of the 1ſt payable Debt, which Quote add to the Product of the two Times, and call this Sum the 2d Number found; Subtract this 2d Number found from $\frac{1}{4}$ of the Square of the 1ſt Number found; and out of the Difference extract the Square Root; which Root being added to or ſubtracted from the half of the 1ſt Number found, the Sum or the Difference will be the Time ſought, in the ſame Denomination: And to know which is the Anſwer, you muſt apply both according to the Conditions of the Queſtion. Thus, If you take the Sum, then, if that is a Time greater than the Time to the laſt payable Debt, the Difference will be the Time ſought: Or if you take the Difference, and that be leſs than the Time to the Term of the firſt payable Debt, the Sum is the Time ſought. But having tried either, and found it betwixt the Terms of Payment of the two Debts, you may try if the other does not caſt it beyond the laſt, or within the firſt Term; for in this Caſe, that which was firſt tried is the Anſwer: But if both give Times betwixt the two given Terms, then you muſt examine which of them will make an Equality of Intereſt and Diſcompt (in the Manner above explained).

Example: There is 100 *l.* payable 1 Year hence, and 105 *l.* payable 3 Years hence: What is the equated Time, allowing Simple Intereſt at 5 *per Cent.* per Annum? *Anſwer*: 2 Years.

The *Operation* according to the *Rule*.

Debts $\left\{\begin{array}{l}100\,l.\\105\end{array}\right.$	Intereſt of 100 *l.* for 1 Year is 5 *l.*	One Debt 100 Its Time 1	The other Debt 105 Its Time 3
Sum 205		Product 100	Product 315

205 divided by 5, the Quote, is 41
which added to the Sum of the Times } 4
The Sum is 45
the 1ſt Number found.

The Sum of theſe Products is 415, which divided by 5, quotes 83, which added to 3 (the Product of the two Times 1 and 3) the Sum is 86, the 2d Number found.

Then the Square of 45 (the 1ſt Number found) is 2025, whoſe $\frac{1}{4}$ Part is 506.25; from which taking 86 (the 2d Number found) the Difference is 420.25, whoſe Square Root is 20.5 Now this being added to 22.5 (the $\frac{1}{2}$ of the 1ſt Number found) the Sum is 43, which cannot be the Anſwer of the Queſtion, becauſe it is greater than the Diſtance of the laſt Term in the Queſtion; wherefore I take the Difference of 20.5 and 22.5; it is 2 Years, the true Anſwer: Which we prove alſo by Application: For this Time being exactly in the middle betwixt the two given Times, the Intereſt of 100 *l.* for 1 Year is equal to the Diſcompt of 105 *l.* for 1 Year; each of them being 5 *l.*

2. If there are more Debts than two, Find an equated Time for the two that are firſt payable: then conſider their Sum as a Debt payable at that equated Time, and find another equated Time for that Debt, and the next of the given Debts, and ſo on, through them all.

For the *Demonſtration* of the 1ſt Part of this *Rule*, thoſe who have not ſtudied the Method of Demonſtration uſed in the preceding Parts of this Work cannot poſſibly underſtand

ſtand it; and to thoſe who have, I need only lay down the general Steps of the Demonſtration, and leave Particulars to themſelves. Thus,

Let the Debt firſt payable be called	d	The laſt payable Debt	D
The Diſtance of its Term of Payment	t	The Diſtance of its Term	T

The Diſtance of the equated Time - - - - - - - x
The Rate of Intereſt, or 1 Year's Intereſt of 1 *l.* r

The Diſtaance of the T ne t and x is $x-t$
The Diſtance of the Time T and x is $T-x$ } For x lies betwixt them.

Then the Intereſt of d, for the Time $x-t$, is $dr\times\overline{x-t}$, or $drx-drt$, (dr being 1 Year's Intereſt of d) and $\frac{DTr-Drx}{1+Tr-rx}$ is the the Diſcompt of D for the Time $T-x$ ($Tr-rx$ being the Intereſt of 1 *l.* for that Time, which is conſequently the *Diſcompt* of $1+Tr-rx$ for the ſame Time): Wherefore we have, from the Nature of the Queſtion, this *Equation, viz.* $drx-drt=\frac{DTr-Drx}{1+Tr-rx}$; which being reduced according to the Common Rules of *Algebra*, comes at laſt to this Equation, $\overline{T+t+\frac{D+d}{dr}}\times x-x^2=\frac{DT+dt}{dr}+Tt$. Now $T+t+\frac{D+d}{dr}$ is what in the *Rule* I have called the 1ſt Number found, and $\frac{DT+dt}{dr}+Tt$ the 2d Number found: Call the 1ſt, a, and the 2d, s, then will the Equation ſtand thus, $ax-x^2=s$; and, by *Problem* 6. *Chap.* 2. *Book* III. we have $x=\frac{a}{2}+$ or $-\overline{\frac{aa}{4}-s}\big|^{\frac{1}{2}}$, which is the preſent *Rule.*

Some will be ready to think I have taken too much Pains about a Queſtion of no great Moment in Buſineſs; and that in common Affairs any of the *Rules* may do without any conſiderable Error. I do freely own the laſt Part: Yet I believe what is done may prove a very uſeful Exerciſe for a Student; and in all Caſes I think Truth is worth the knowing.

ADDITION to *Chap.* 8. *Book* VI.

Of the Alligation, *or* Mixture *of Bodies, with reſpect to their Specifick Gravities.*

DEFIN. WHEN of Bodies of different kinds of Matter there are equal *Maſſes* or *Bulks*, for Example, a Sphere of the ſame Diameter (or an equal Cube) their different Weights are called their *Specifick Gravities:* And tho' the Weight is greater or leſſer, as the Bulk is, yet being ſo always proportionably, the Weights under any equal Bulk whatever are called the *Specifick Gravities*; ſo that the more ſtrict Definition is, the *Proportion of Weights under the ſame Bulk.*

Some Bodies of different Specifick Gravities are capable of being mixt and incorporated with one another, as are ſome Liquors and Metals; and as the Mixture muſt neceſſarily make the compound Body of a Specifick Gravity different from that of all, or all but one, of the Bodies mixt; ſo the ſame Problems ariſe about their *Mixture*, with reſpect to their Specifick Gravities; and the Proportion of the Quantities mixt, as before with reſpect to the

the Quantities and Prices of Liquors, and other things; and which are to be solved by the same Methods of Operation.

But in order to understand the Application, there is one Proposition to be first explain'd, which is this, *viz.*

The *Bulks* (or Numbers of cubical Inches) of two Bodies of equal Weight (which I shall call their *Specifick Bulks*), are reciprocally proportional to their Specifick Gravities.

Thus: Call two Species of Bodies A and B; if the Specifick Gravity of A is to that of B, as 1 to 2, then the Bulk of A is to that of B of the same Weight as 2 to 1. For Example; If a Body of a cubical Inch of one Metal is 2 Ounces, and a Body of a cubical Inch of another Metal is 1 Ounce, then it's plain that the Half Bulk of the first will be as heavy as the Whole of the other. Or if the Specifick Gravities are as 5 to 3, then $\frac{3}{5}$ of the first is as heavy as the Whole of the other. And so in any other Proportion.

Hence again; If the Specifick Gravities of 2 or more Bodies are expressed by any Numbers whatever, their *Specifick Bulks* are expressed by the Reciprocals of these Numbers, thus; If the *Specifick Gravities* of 3 Bodies are as 5 . 7 . 8, their *Specifick Bulks* are as $\frac{1}{5} . \frac{1}{7} . \frac{1}{8}$; and if the *Specifick Gravities* are as $\frac{2}{3} . \frac{5}{7} . \frac{3}{4}$, the Bulks are as $\frac{3}{2} . \frac{7}{5} . \frac{4}{3}$; because any two Numbers are proportional with their Reciprocals, taken reciprocally. For the same Reason, however the Reciprocal Bulks are expressed, the Specifick Gravities are expressed by the Reciprocal Numbers.

These things being premised, I shall next apply the Rules of *Alligation* to the *Mixture* of Bodies, with respect to their *Specifick Gravities* (or Bulks, which is in effect the same thing), thus:

Case 1. If there are given the *Specifick Bulks* of several Bodies (or *Specifick Gravities*, by which the Bulks may be found), together with the Quantities or Weights of each put into a Mixture; to find the *Specifick Bulk*, or *Gravity* of the *Mixture*; it is done after the manner of *Case* 1. *Chap.* 8. thus:

Rule. Find the total Bulk (or Numbers of cubical Inches) of each Quantity, by the Specifick Bulk; add all these Bulks into one Sum, and also the Quantities taken, and say, as the total Quantity to the total Bulk, so is an Unit of the Denomination by which the Specifick Bulks are expressed, to the Specifick Bulk sought.

Example: Suppose 10 Ounces of Metal, whose Specifick Bulk is 3 (*i.e.* 3 cubical Inches to 1 Ounce) are mixed with 14 Ounces of another whose Specifick Bulk is 2: What is the Specifick Bulk of the Mixture? *Answer*, $\frac{12}{29}$: Which I find thus: If 1 Ounce makes 3 cubical Inches; then (by the *Rule of Three*) 10 Ounces make 30; and if 1 Ounce makes 2, then 14 Ounce make 28: and, lastly, if 24 (*i.e.* 10 and 14) Ounces make 58 (*i.e.* 30 and 28) cubical Inches, 1 Ounce makes $\frac{12}{29}$. Then the Specifick Gravities of the Metals that are mixed, and of the Compound, are $\frac{1}{10}, \frac{1}{14}, \frac{29}{12}$.

The *Reason* of the *Operation* is obvious.

Observe, If the given Quantities have all one Denomination, or are reduced to one, whatever that is, they may then be conceived to be all of that Denomination to which the *Specifick Bulks* are supposed to refer; since the Proportion is still the same: So hereby the Work becomes easier; for we have nothing to do but multiply the Quantity of each Simple (after they are reduced to one Denomination) by its Specifick Bulk, and divide the Sum of the Products by the Sum of the Quantities; as it is in the preceding *Example*.

Case 2. Having the *Specifick Bulks* (or Gravities by which the Bulks are found) of several Bodies to be mixt, and the Specifick Bulk (or Gravity) to which a Mixture of these Bodies is to be reduced; To find the proportional Quantities to be taken of each, that the Mixture may bear the given Rate of Specifick Bulk (or Gravity).

Rule. Take the given Specifick Bulks of the Bodies to be mixt, link them together, then take and place their Differences from the Specifick Bulk of the Mixture, the same way as taught in Alligation alternate, and you have the Answer. *Exam-*

Cub. Inch.		lb Weight.
2		2
	3	
5		1

Example. Suppose two Metals, whereof the one has 2 cubical Inches to 1 lb Weight, and the other 5; What Proportion of Weight of each must be taken to make the Mixture 3 cubical Inches to 1 lb *Ans.* 2 lb of the 1st to every 1 of the other, as in the Margin.

To shew the *Reason* of this, we must premise this Truth, *viz.* That of three Numbers, if the Difference of the least Extreme and middle Term is multiplied by the greater Extreme; and the Difference of the greater Extreme and middle Term is multiplied by the lesser Extreme; the Sum of their Products is equal to the Product of the Sum of the two Differences by the middle Term; as is plain in the annexed Scheme, where the three Numbers being a, b, c, their Differences, according to the *Rule*, are $c-b$, and $b-a$, whose Sum is $c-a$; then the Products mentioned are $ac-ab$, and $bc-ac$, whose Sum is $bc-ab$ equal to $c-a$ multiplied by b.

$$\begin{array}{ll} a & \times\overline{c-b}=ac-ab \\ b & \\ c & \times\overline{b-a}=bc-ac \\ b & \times\overline{c-a}=bc-ab \end{array}$$

Now to apply this to the *Demonstration* of the *Rule*, Consider in the above *Example*, that 2 lb, whereof each is 2 cubical Inches, make 4 cubical Inches; and 1 lb of 5 cubical Inches, is 5 cubical Inches; the Sum is 9 cubical Inches; so that the Mass of Mixture is 3 lb, and the Bulk is 9 cubical Inches; therefore, the Specifick Bulk, or Bulk of 1 lb of the Mixture is the 3d Part of 9, *viz.* 3 cubical Inches, the proposed Specifick Bulk of the Mixture; which is therefore justly proportioned: Or thus; the Product of the total Weight mixt 3 lb, multiplied by the given Specifick Bulk 3, is 9; equal to the Sum of the Products of the particular Quantities multiplied by their respective Specifick Bulks; therefore the Solution must be true; and, by the universal Truth premised, it must be so in all Cases of two Bodies mixt; and consequently of any Number.

Observe, 1. The same Method of *Demonstration* may be applied to the Mixture of any other things, as Liquors, *&c.*

2. All the same kind of Questions made above with respect to Liquors, may also be made with respect to Specifick Gravities, or Bulks of Bodies.

3. By this Rule is the famous Question solved about *Hiero*, King of *Syracuse*'s Crown: He gave a certain Quantity of pure Gold to make a Crown; but suspecting the Goldsmith had mixt Silver with it, he desired *Archimedes* to discover it if possible; who did it by this Means: When he went into a bathing Tub, he reflected that every Body immersed in Water must put as much Water out of its Place as is equal in Bulk to it self: Therefore he took a Quantity of pure Gold, and another of Silver, each of the Weight of the Crown; or, as some say, caused a Crown to be made of pure Gold, and another of Silver, each of the same Weight with the 1st Crown, and measured their Specifick Bulks by the Bulks of the Quantities of Water put out of its Place by the Immersion of each of the 3 Crowns (or of the suspected Crown, and each of the Masses of pure Gold and Silver; which would put out the same Quantity of Water, whatever Shape they were in) and by Comparison of these 3 Specifick Bulks, he found how much Gold and Silver was in the mixt Crown; which may be done after the manner of *Qu.* 3. *Page* 569. thus: We have the Specifick Bulks of the Mixture, and of the two Simples; by which, finding the Proportion of Weights that brings them to that Specifick Bulk, if their Sum is equal to the Weight of the mixt Crown, then the Proportion of Gold and Silver in it is discover'd; by which the Gold and Silver separately are easily found: And if the Sum is not the same as the Weight of the Crown, yet that Weight being divided into the same Proportion as the Parts of that Sum, solves the Question.

FINIS.

ERRATA.

Page	Line	For	Read
101.	42.	$a+bc$	$\overline{a+bc}$
111.	10.	$\frac{1}{54}$	$\frac{18}{54}$
131.	8.	$\frac{3}{5}$	$\frac{5}{3}$
239.	12.	$-\frac{2a}{2}$	$=\frac{2a}{2}$
288.	45.	*Theor.* IV.	*Theor.* VII.
383.	34.	after *oddly even only*, add, *above* 2.	
416.	22.	$-\frac{n^{m+2}+n^{m+1}}{2}$	$-\frac{n^{m+2}+n^{m+1}}{2}$
424.	20.	3	8
472.	11.	$10m\times\overline{10m-1}$	$10^m\div\overline{10^m-1}$
ib.	ib.	$l\times l^n$	*Extreme.*
486.	11.	2d or 3d	2 or 3.
ib.	14.	$a^n\times a^m$	$a^m\times a^m$
ib.	ib.	$a^{2m}\div a$	$a^{2m}\div a^n$
488.	31.	$\overline{n-1a}$	$\overline{n-1}\times a$
ib.	40.	$n-10$	$\overline{n-10}$
489.		To the Scheme of Numbers, after *Line* 18. add this Line of Numbers, *viz.* 0 : 1 : 2 : 3 : 4 : 5 : 6 : 7 : 8	
499.		In the Scheme, after *Line* 25. for 2.9332209, *read* $\bar{2}$.9332209.	
514. 515.	25. 13.	$\times B-\overline{A-1}$	$\times\underline{B-\overline{A-1}}$
ib.	ib.	$\times B-A$	$\times\overline{B-A}$
ib.	33.	$A+1$	$\underline{A+1}$
517.	14.	$\frac{n-2}{1}$	$\frac{n-1}{2}$
545.	2.	3 oz. 14 dr.	1205 ℔ 3 oz. 14 dr.
551.	21.	18148	18338
559.	12.	*what*	*which*
616.	35.	*Periods*	*Pounds.*

In the *Preface*, at the Foot of *Page* ix. for *Chap.* 11. read *Chap.* 10. § 5.